

Gradient Marker: A Universal Wave Pattern in an Inhomogeneous Continuum

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Wave transport in a media with a slow spatial gradient of its characteristics is found to exhibit a universal wave pattern (“gradient marker”) in a vicinity of the maxima or minima of the gradient. The pattern is common for optics, quantum mechanics, and any other propagation governed by the same wave equation. Derived analytically, it has an elegantly simple yet nontrivial profile found in perfect agreement with numerical simulations for specific examples. We also find resonant states in continuum in the case of quantum wells, and formulated criteria for their existence.

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Wave patterns in inhomogeneous media or confining structures are of great interest to quantum mechanics (QM), optics and electrodynamics, acoustics, hydrodynamics, and chemistry. Examples include wave packets in atoms [1], Ghladny patterns in acoustics, EM resonator and waveguide modes [2], Anderson localization in disordered systems [3], soliton formation [4] due to nonlinearity, including atomic solitons in bosonic gas [5], as well as giant waves near caustics [6], waves in chemical reactions [7], dark-soliton grids [8], “scars” in “quantum billiard” [9], “quantum carpets” in QM potentials [10], nanostratification of local field in finite lattices [11], etc. In all of those, the presence of multimodes or a broadband spectrum is a prerequisite for interference and pattern formation in inhomogeneous or confining structures.

In this Letter we show, however, that a localized wave pattern—an immobile single-cycle intensity profile—can emerge in a single-mode wave in the vicinity of a minimum or maximum of the gradient of a QM potential or optical refractive index. The phenomenon is universal for both optics and QM, and for any other propagation described by a wave equation, (1) below. What makes it unusual is that it emerges in media with no potential wells and only a smooth inhomogeneity yielding no reflection—and is originated by a purely traveling wave with apparently no other modes to interfere with. We found, however, that this wave here generates a cotraveling but localized “satellite” of slightly different phase and amplitude resulting in “self-interference.” The wave ideally is not trapped and carries its momentum and energy flux unchanged through the area. To a degree, the pattern mimics a 2nd order spatial derivative of the refractive index (or potential function); it would be natural to call it a “gradient (*G*) marker”. In QM it may be most pronounced for an above-barrier propagation of an electron in continuum over smoothly varying potential; in solid state it might emerge above the critical temperature for the Anderson localization to vanish. Even for a potential well, when the energy of electrons exceeds the ionization potential and there is no trapping, the *G* markers emerge as the main nonresonant localized feature.

To demonstrate the effect and elucidate analytical results (to be compared with numerical simulations) we consider a 1D case written, for the sake of compactness, in “optical” terms, using space-varying refractive index $n(x)$; yet we consistently “translate” all the effects and approaches into QM terms. A 1D spatial dynamics of an ω -monochromatic plane wave with linearly polarized electrical field $\vec{E} = \hat{e}_p E(x) \exp(-i\omega t) + \text{c.c.}$, propagating in the x axis (here $\hat{e}_p \perp \hat{e}_x$ is a polarization unity vector), is governed by the wave equation

$$E'' + n^2(\xi)E = 0; \quad \xi = xk_0, \quad (1)$$

where $k_0 = \omega/c = 2\pi/\lambda_0$, and “prime” stands for $d/d\xi$. (For \vec{H} field, $\hat{e}_x \perp \vec{H} \perp \hat{e}_p$, one has $H = -iE'$ in non-magnetic materials; for a traveling wave, $|H| = n|E|$, if $n = \text{const.}$) In QM terms, this corresponds to 1D scattering of a particle in continuum by a potential $U(x)$, with E replaced by a wave function, ψ , of a particle, H —by $(-i\psi') = p_Q/k_0\hbar$, n —by $p_C/k_0\hbar$, where p_Q and $p_C = \sqrt{2m[\mathbf{E}_0 - U(x)]}$ are its quantum and classical momenta respectively, and \mathbf{E}_0 —full energy. We will consider only the case $n^2 > 0$, where one can attain a no-reflection mode of the main interest to us here; otherwise, with $n^2(\xi)$ crossing zero, the system may exhibit a full reflection characterized by an Airy function as, e.g., near a turning point in QM [12], or a critical point in plasma [2], or caustics in optics and water waves [6].

Equation (1) is ubiquitous in physics and engineering. Since few known functions $n(\xi)$ allow for analytical solutions, numerical simulations and/or approximate analytical solutions in general have to be used. Of the most interest to us here will be the limit of adiabatically *slow* variation in space, when gradient parameter $\mu \sim (k_0 L n_{\min})^{-1}$, where L is a spatial scale of inhomogeneity, is small, $\mu \ll 1$, which corresponds to a quasiclassical case in QM. The reflectivity R in this case vanishes as $R = O(e^{-A/\mu})$ [12,13], where $A = O(1)$ (usually $A > 1$), and reflection can be neglected by a large margin. A solution is provided then by a WKB approximation [12] as traveling waves, $C^{(\pm)}$

$\exp(\pm i \int n d\xi)/n^{1/2}$. Considering, e.g., a forward wave, and setting $n' \rightarrow 0$ at $|\xi| \rightarrow \infty$, where we normalize its intensity by setting $n|E|_\infty^2 = 1$, we look for the next approximation as a perturbed WKB solution

$$E = [1 + \Delta(\xi)]e^{i \int n d\xi}/n^{1/2} \quad \text{with} \quad \Delta = \gamma + i\beta, \quad (2)$$

where $\Delta(\xi)$ ($|\Delta|^2 \ll 1$) is a slow-varying complex function, $\Delta \rightarrow 0$ at $|\xi| \rightarrow \infty$, γ and β -real; as we will see later on, $|\gamma|_{\max} \sim |\beta|_{\max} = O(\mu^2)$. Using ansatz (2) in Eq. (1), setting real and imaginary parts of the sum of all the perturbations terms to zero, and collecting the terms of lower order in μ in each one of them, we obtain for the real part an equation consisting of $O(\mu^2)$ terms

$$\beta' = -(n'/n^{3/2})'/4n^{1/2} \quad (3)$$

and for the imaginary part—an equation consisting of $O(\mu^3)$ terms, the integration of which yields $\gamma = -(\beta^2 + \beta'/n)/2$, where we set the integration constant to zero due to the above condition $\Delta_{\pm\infty} = 0$. [It is worth noting that in the end, all the terms with β^2 get canceled, so there is no need for further integration of Eq. (3).] We can finally arrive at a G -marker intensity by calculating the perturbation, $\delta I(\xi) \equiv I - 1$, of the normalized field intensity $I \equiv n|E|^2 = (1 + \gamma)^2 + \beta^2 \approx 1 + 2\gamma + \beta^2$, retaining the terms lowest in μ , and obtaining to $o(\mu^2)$:

$$\delta I(\xi) = -\beta'/n = (n'/n^{3/2})'/4n^{3/2}. \quad (4)$$

In the vicinity of a gradient peak, $\delta I(\xi)$ makes an asymmetric single-cycle shape, with its middle point shifted by $O(k_0 L)$ toward the area with a lower refractive index (or higher potential); its higher (and positive at that) peak is also located in the same area, see Figs. 1 and 2. One can see that $\delta I(\xi)$ more or less mimics a second derivative of n . Equation (4) can also be obtained via quasiclassical approximation in QM [12], whereby one has to search for high-order corrections for the phase of ψ as a function of the classical momentum p_C , after which it has to be translated into correction to intensity.

How far the asymptotic result (4) can be pushed beyond the limit $\mu \ll 1$, and what is a critical $\mu_{cr} = O(1)$, can be explored only by numerical simulations, which also helps to reveal a nature of a small parameter μ (which appears to be substantially different from a standard $(|n'|/n^2)_{\max} \ll 1$ [14]). Before comparing Eq. (4) to numerical simulations for specific profiles $n(\xi)$ and various μ , let us make sure it conforms to the conservation of EM energy flux, i.e., (time-averaged) magnitude, \bar{S} , of the Poynting vector, $\vec{S} = \vec{E} \times \vec{H}/2$ (in Abraham's form) in the general case. Writing $S = (Ee^{-i\omega t} + \text{c.c.})(He^{i\omega t} + \text{c.c.})/2$, and t -averaging it, which amounts here to omitting terms with $e^{\pm 2i\omega t}$, we have $\bar{S} = \text{Re}(EH^*) = \text{Re}(iE^{*'}E)$. In QM terms, it corresponds to

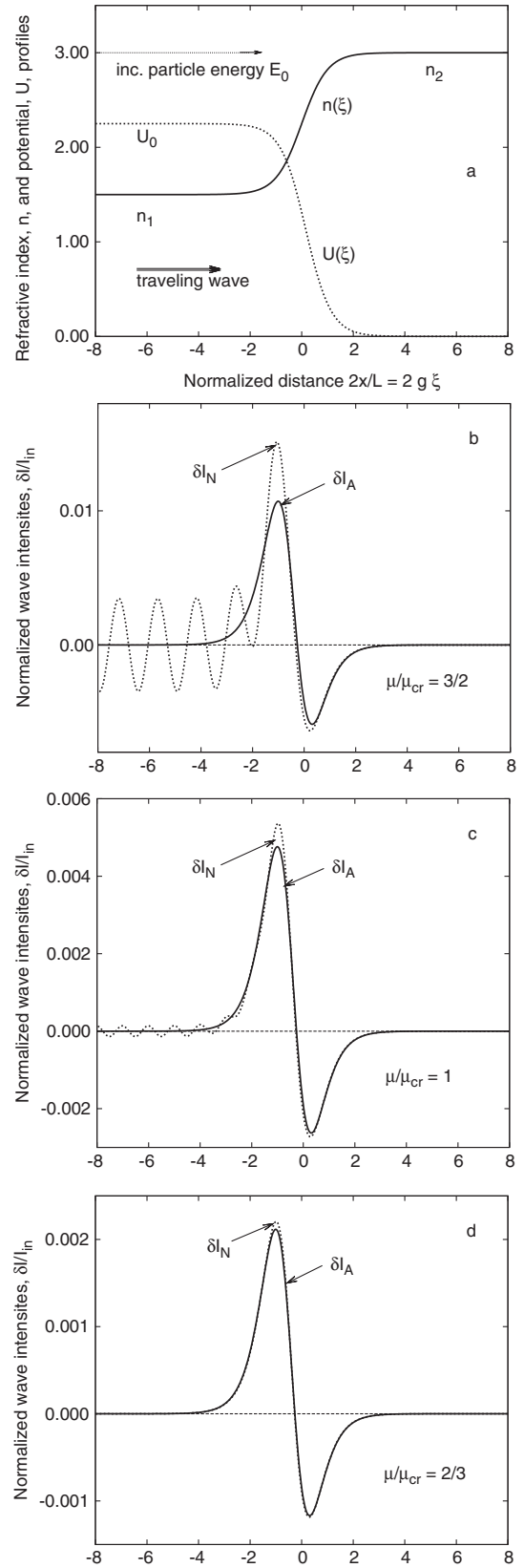


FIG. 1. (a) Refractive index, n (and potential U) soft-step spatial profiles, $n_1 = 1.5$ and $n_2 = 3$, $\mu_{cr} \approx 0.24$; (b)–(d) G -marker intensity, δI , vs distance $x/2L$ for various parameters μ ; curves: δI_N -numerical, and δI_A -analytical, Eq. (4).

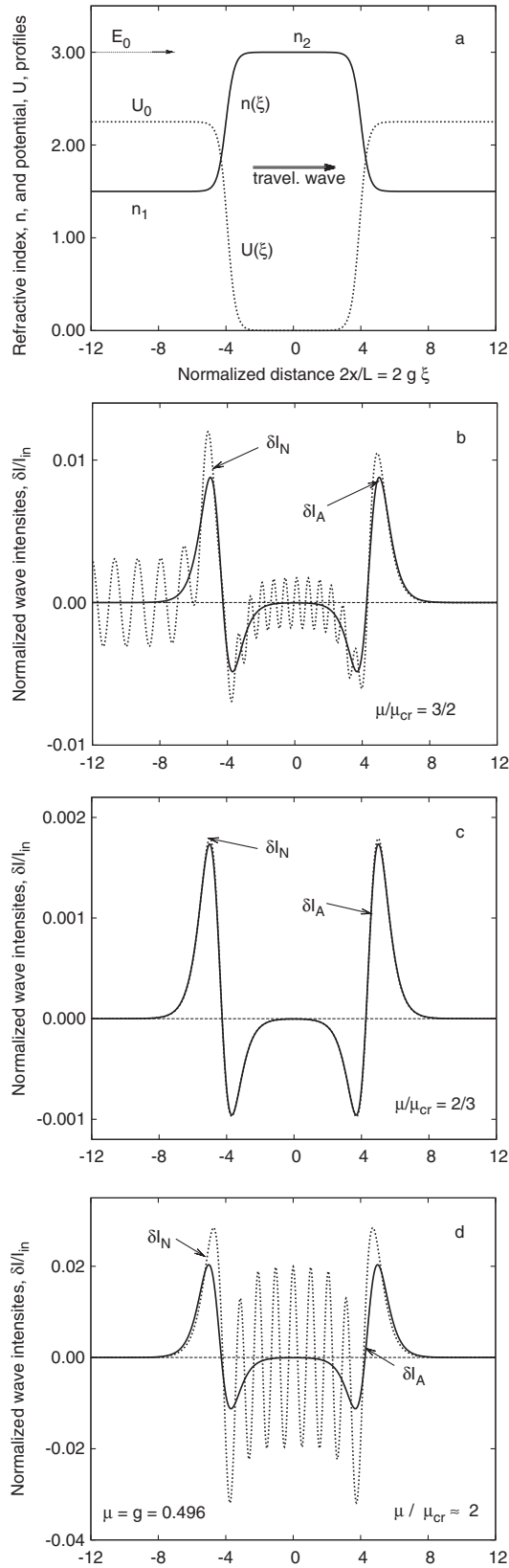


FIG. 2. (a) Refractive index plateau, n (and potential well U) profiles, with n_1 , n_2 , and μ_{cr} same as in Fig. 1; (b,c) δI , vs $x/2L$ for $\mu > \mu_{cr}$ (b), and $\mu < \mu_{cr}$ (c), with δI_N and δI_A as in Fig. 1; (d) resonant state in continuum (see the text).

mathematical expectation of a particle momentum, $\langle \psi | \hat{p}_Q | \psi \rangle$, $\hat{p}_Q = -i\hbar d/dx$. Using Eq. (2) and retaining the terms of the lowest (2nd here) order in μ , we have $\bar{S} = I + \beta'/n$; due to Eq. (4) it confirms that $\bar{S} = 1 = inv$ to $o(\mu^2)$.

For numerical simulations of Eq. (1) with an arbitrary profile $n(\xi)$ and arbitrary μ , we broke it into two 1st order (Maxwell, in EM-case) equations: $E' = iH$; $H' = in^2E$. To model a “soft step” $n(\xi)$, we use a function (Fig. 1(a)):

$$n(x) = n_1 + (n_2 - n_1)[1 + \tanh(2x/L)]/2 \quad (5)$$

with controllable L , n_i , and its gradient parameter as

$$\mu = g(n_1^{-1} + n_2^{-1}); \quad g = (k_0L)^{-1} = \lambda_0/2\pi L \quad (6)$$

[14]; the spectral dispersion of n_i can be safely ignored here. The calculations with an arbitrary μ are done by using a multipoint algorithm and a “reverse propagation” mode, whereby we start at $\xi \gg g^{-1}$, postulate that only one (transmitted) field remains there, $E_\infty \rightarrow \exp(in_2\xi)/\sqrt{n_2}$, (the Sommerfeld’s condition), and then go backward, till reaching a symmetrically located area in front of the gradient, $\xi \ll -g^{-1}$, where we record the intensity of an incident wave, $I_{in} = n_1|E + H/n_1|^2/4$, and then normalize all the stored intensities by I_{in} [15]. The precision of calculations is checked against the deviation at the incidence; typically it was better than 10^{-6} .

The retroreflection from the gradient area is strongly suppressed and the G marker is well emphasized (see, e.g., Figs. 1(c) and 1(d)) provided that $\mu < \mu_{cr}$, where parameter μ_{cr} was found by us to be almost universal, $\mu_{cr} \approx (2\pi)^{-1}$ at $r_n \equiv n_1/n_2 + n_2/n_1 \gg 1$, and slightly increasing to $\mu_{cr} \approx 1/4$ near $n_1 \sim n_2$. The highest (positive) G -marker peaks, δI_{max} , can easily reach a few percent of the intensity, I , especially at $r_n \gg 1$. In the case of a “shallow” soft step, $|n_1 - n_2| \ll \bar{n}/2$, $\bar{n} = (n_1 + n_2)/2$ [in QM this would correspond to a kinetic energy E_0 much higher than the drop of potential, U_0 , $E_0/U_0 \sim \bar{n}/|n_1 - n_2| \gg 1$], the maxima and minima of δI are almost of the same magnitude,

$$|\delta I_M| \approx 2g^2|n_1 - n_2|/(5^{3/2}\bar{n}^4) \quad (7)$$

and located at $x \approx \pm L/4$. (In general, the parallel between optics and QM can be guided by the relationship $U_0/E_0 = 1 - \min[(n_1^2/n_2^2), (n_2^2/n_1^2)]$.)

Figures 1(b)–1(d) for the case of $n_1 = 1.5$, $n_2 = 3$, $\mu_{cr} = 0.24$, show numerical simulations of spatial dynamics of δI_N , converging amazingly fast to an asymptotic analytical result for a G -marker intensity, δI_A , Eq. (4), as soon as $\mu \leq \mu_{cr}$. Figure 1(b) with $\mu/\mu_{cr} = 3/2$ shows a residual reflection giving rise to an oscillating structure (partial standing wave), comparable in its amplitude to a G marker, while Fig. 1(c) ($\mu = \mu_{cr}$) depicts a distinct and strong G marker formed even at $L \approx \lambda_0/2$. Finally, an

inhomogeneity with $L = \lambda_0$ [Fig. 1(d), $\mu/\mu_{\text{cr}} = 2/3$] is sufficient to produce a very clean G marker.

We move now to investigate G -marker formation by a potential well (or a refractive index plateau) by modeling it with an “up-and-down” double step, Fig. 2(a):

$$n(x) = n_1 + (n_2 - n_1)(T_+ - T_-)/2 \tanh(D/L), \quad (8)$$

where $T_{\pm} = \tanh[(2x \pm D)/L]$, and D is a controllable spacing between the steps. For $D \ll L$, it becomes $n(x) = n_1 + (n_2 - n_1)/\cosh^2(2x/L)$, but for our purposes here we choose a more boxlike well, $D/L = 8$, which has μ defined by Eq. (6), and the same μ_{cr} as for a soft step (5). As expected, both walls form G markers symmetric to each other, Figs. 1(b) and 1(c), so that to form a G marker it does not matter which way a wave is arriving—from the lower index or from the higher one. At $\mu > \mu_{\text{cr}}$ one can see some oscillations, same as for a single wall in Fig. 1(b) for the same μ , and ideally clean G markers for $\mu < \mu_{\text{cr}}$, similar to Fig. 1(d) for the same μ .

The major difference here comes, however, in the area $\mu > \mu_{\text{cr}}$. Here, at a certain (countable) set of points in the continuum, while there are strong oscillations within a potential well, which indicates a significant wave reflection between G markers, there is no reflection from the entire potential well, see Fig. 2(d). That confined partially standing wave is a signature of a resonant state in a finite-depth quantum well with rigid walls, most known in the case of a finite rectangular box. Figure 2(d) depicts one of those states with $\mathbf{E}_0/U_0 = 4/3$. The condition for them to emerge above a quantum well is a significant rigidity of the well’s walls, $\mu > \mu_{\text{cr}}$. In the limit $\mu \gg \mu_{\text{cr}}$, their energies in the continuum coincide with those of a finite box, or in turn—with the eigenstates of a box with infinitely-high walls, $\mathbf{E}_N = (N\hbar\pi)^2/2mD^2$, where N is a natural number, provided that $\mathbf{E}_N > U_0$. In optics terms, they correspond to full-transmission resonances of a Fabry-Pierrot resonator with semitransparent mirrors. In the solid state, these states may reveal themselves during a δ -kick field ionization via production of spatially stratified bunches in photoelectron current, whose kinetic energies coincide with those of the resonant states [16].

Potential uses or applications of 1D (or almost 1D) G markers can be envisioned, such as (a) observation of quantum “traces” in continuum, i.e., beyond “quantum carpets” [9] in potential wells, (b) detection and control of slight changes of optical fiber parameters [17], (c) the diagnostics of cold underdense plasma, (d) medical surface-wave ultrasound tomography, (e) detection of the movement of near-shelf profiles of the bottom of oceans and rivers by space- or air-borne photography of the patterns of wind-driven gravitation waves, as well as (f) contour detection and tracing of submerged large moving manmade objects or whales in the ocean.

A 2D and 3D expansion of the theory may need to be developed for other potential applications of G markers

such as (g) the “tomography” of quantum landscape in disordered solid-state at above-critical temperature [3], (h) a bulk tomography of opaque fluids (e.g., oil or muddy water) by using nonpenetrating surface EM or acoustic waves, or of solid-state bodies (e.g., in “introvision” of computer chips, or lacunas in blobs of metallic alloys or glass), as well as (i) in plasma- and astrophysics.

In conclusion, we predicted the formation of a universal feature in wave transport in inhomogeneous media—a standing single-cycle spatial modulation of wave intensity—gradient marker—located in closed vicinity of the maximum or minimum of a gradient of refractive index or potential function. We found a critical condition for a G marker to be resolved on the background of residual reflection. In the presence of a trapping potential, we also found resonant states or modes in the continuum at the energies above photoionization and formulated the condition of those modes to exist when G markers are not dominant.

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