

Isolas in Four-Wave Mixing Optical Bistability

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Abstract—Optical bistability based on the cross-interaction of two counterpropagating plane waves in a Kerr-like medium is considered. It is shown that the “input-output” characteristics of such a system have not only multivalued hysteretic response (which was demonstrated by Winful and Marburger in 1980), but also multiple isolated branches (so called isolas) with nonhysteretic jumps from these branches to the main curve (but not vice versa). The isolas were found both in the case in which the problem can be solved analytically (particularly for electrostriction mechanism of optical nonlinearity) and in the general case of arbitrary light-induced anisotropy.

INTRODUCTION

WINFUL and Marburger [1] have first proposed an optical bistability in collinear degenerate four-wave mixing. This effect offers a very interesting and fundamental principle of optical bistability [2] since it is based on the cross-interaction of two counterpropagating plane waves in a nonlinear material with no other electrodynamic components involved. Like some other principles of optical bistability [3]–[7], it does not require any cavity or Fabry–Perot resonator (as opposed to many conventional bistable effects [2]); however it differs from those principles in using neither reflection at nonlinear interface [3], [4] nor transversal cross-interaction of counterpropagating beams with limited cross-section (such as cross-self-focusing bistability [5] or any other cross-self-action bistability [6], [7]). The closely related effects have also been discussed in [8] and [9], and experimental observation of optical bistability due to six-photon mixing was first reported in [10].

In this paper we show that the bistability based on degenerate four-wave mixing can demonstrate quite interesting and unusual (for optical bistability) features such as multistable but nonhysteretic behavior (e.g., existence of semiinfinite multiple branches of solutions, when the numbers of “on” and “off” jumps are different), and, what is most interesting, existence of isolated limited branches of solutions (which are referred to as isolas in the theory of nonlinear systems in various fields [11]), see Sections B and C, from which the system can “jump-off,” but could not “jump on” (in a quasi-steady-state regime). The latter solutions apparently correspond to some internal nonlinear self-induced resonances in four-wave mixing, and, as such, deserve closer attention in further re-

search since they probably play an important role in the stability of the system, and may result in a strongly pronounced chaotic motion in the range of possible instability.

In order to demonstrate these features, we first use the special cases of the anisotropic characteristics of Kerr-nonlinearity (specifically $B = 0.5$, and $B = 0$, see below), which provide an opportunity to get a thorough analytical solution for the wave equations (Section B); in the Section C we demonstrate that the isolated branches exist for more general case (for concrete calculations, we use the case $B = 0.2$ since it yields simpler elliptic integrals). All the calculations are based on the equations for the nonlinearly intercoupled circular components (Section A).

A. Wave Equations and Their Invariants

We consider two optical beams with the same frequency that are counterpropagating along the z axis, such that the total (complex) electric field is

$$\vec{E} = [\vec{E}_1 \exp(ikz) + \vec{E}_2 \exp(-ikz)] e^{-i\omega t}. \quad (1.1)$$

(The total real field is $(\vec{E} + \vec{E}^*)/2$ where an asterisk denotes complex conjugation.) Here \vec{E}_1 and \vec{E}_2 are envelope vectors of forward and backward waves respectively; both of them are complex, slowly varying with respect to e^{ikz} , and arbitrary polarized. Based on the results [12], [13], it could be shown that for an arbitrary type of third-order nonlinear susceptibility, the nonlinear (complex) component of electric displacement \vec{D}^{NL} can be expressed as

$$\vec{D}^{\text{NL}} = \epsilon_o \chi [A \vec{E}(\vec{E} \cdot \vec{E}^*) + B \vec{E}^*(\vec{E} \cdot \vec{E})] \quad (1.2)$$

where ϵ_o is a linear electric permittivity of medium, χ is a constant of nonlinear interaction for a single linearly polarized plane wave, A and B are dimensionless constants dependent upon the particular mechanism of nonlinearity. In a lossless medium, A and B are real quantities related by $A + B = 1$. In the case of the pure Kerr-effect, in liquids [14]–[17] (e.g., CS_2) $B/A = 3$, i.e., $A = \frac{1}{4}$, $B = \frac{3}{4}$; in crystals [12], [13], [15] $B/A = \frac{1}{2}$, i.e., $A = \frac{2}{3}$, $B = \frac{1}{3}$, and for the electrostriction [14], [15] $A = 1$, $B = 0$. Therefore, the coefficient B may vary significantly. By substituting (1.1) into (1.2), using the Maxwell equations, and assuming slowly varying envelopes (as well as neglecting the generation of higher order harmonics), one can get the equations [18] for both wave envelopes \vec{E}_1 and \vec{E}_2 as follows:

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$$\begin{aligned} \frac{-2i}{\chi k} \cdot \frac{d\vec{E}_1}{dz} = & A[\vec{E}_1(I_1 + I_2) + \vec{E}_2(\vec{E}_1 \cdot \vec{E}_2^*)] \\ & + B[2\vec{E}_2^*(\vec{E}_1 \cdot \vec{E}_2) + \vec{E}_1^*(\vec{E}_1 \cdot \vec{E}_1)]; \end{aligned} \quad (1.3a)$$

$$\begin{aligned} \frac{2i}{\chi k} \cdot \frac{d\vec{E}_2}{dz} = & A[\vec{E}_2(I_1 + I_2) + \vec{E}_1(\vec{E}_1 \cdot \vec{E}_2^*)] \\ & + B[2\vec{E}_1^*(\vec{E}_1 \cdot \vec{E}_2) + \vec{E}_2^*(\vec{E}_2 \cdot \vec{E}_2)]; \end{aligned} \quad (1.3b)$$

where $I_1 = |E_1|^2$ and $I_2 = |E_2|^2$ are respective intensities of both of the waves. The solution of (1.3) must satisfy boundary conditions at both of the faces of a nonlinear layer of the thickness L :

$$\vec{E}_1(z=0) = \vec{E}_{10}; \quad \vec{E}_2(z=L) = \vec{E}_{2L}. \quad (1.4)$$

In the case $\vec{E}_2 = 0$ (or $\vec{E}_1 = 0$), (1.3) determines one-directional self-interaction of a plane wave, in particular, light-induced gyrotropy, i.e., self-rotating of the main axis of elliptic polarization if the incident wave is slightly elliptically polarized [12]–[17]. When both of these vectors \vec{E}_1 and \vec{E}_2 are linearly polarized and parallel to each other, (1.3) exhibits light-induced nonreciprocity [19] with $\chi_1^{\text{NL}} - \chi_2^{\text{NL}} = \chi(I_2 - I_1)$. There are also three other types of relative arrangements of polarizations for which the polarization configuration of the beams remains unchanged as they propagate through a nonlinear material (so called eigenarrangements [18]): both of the beams are linearly polarized with the vectors \vec{E}_1 and \vec{E}_2 orthogonal to each other, and both of the beams are circularly polarized with \vec{E}_1 and \vec{E}_2 either corotating or counterrotating. All four eigenarrangements have different eigennonreciprocities [18]. In general case of the arbitrary polarization, when both of the intensities I_1 and I_2 are sufficiently large, the system can demonstrate bistability and hysteresis [1].

In order to further investigate (1.3), one has to express them in the scalar form. Usually, the equations for collinear degenerate four-wave mixing [to which (1.3) gives a vectorial counterpart] are written [1], [20] for the coordinate (x and y) components of the respective envelopes \vec{E}_1 and \vec{E}_2 . However, since the nonlinear anisotropy is induced by the field and therefore is not intrinsically tied to any selected directions in the material [which is explicitly seen from (1.3)], we find it more natural and adequate to represent each of the fields \vec{E}_1 and \vec{E}_2 as a sum of two circularly polarized components with the scalar amplitudes. This provides meaningful interpretation of results (in particular, conservation invariants, see below) and makes it easier to find situations in which the simple (e.g., eigenarrangements) or analytical solutions can be found.

Thus, we decompose each of waves \vec{E}_1 and \vec{E}_2 into a sum of two circularly polarized components, rotating in the opposite directions:

$$\vec{E}_j = \frac{1}{\sqrt{2}} [u_j(\hat{e}_x + i\hat{e}_y) + v_j(\hat{e}_x - i\hat{e}_y)]; \quad j = 1, 2 \quad (1.5)$$

where u_j and v_j are complex amplitudes of the respective circular components. Substituting (1.5) into (1.3) and collecting separately the terms with left and right rotating components, one gets the equations for the components u_j and v_j :

$$\begin{aligned} (-1)^j \frac{2iu_j'}{\chi k} = & u_j[(1 - B)(I_\Sigma + |u_{3-j}|^2) \\ & + 2B(|v_1|^2 + |v_2|^2)] \\ & + (1 + B)v_j u_{3-j}^* u_{3-j}; \quad j = 1, 2; \end{aligned} \quad (1.6)$$

$$\begin{aligned} (-1)^j \frac{2iv_j'}{\chi k} = & v_j[(1 - B)(I_\Sigma + |v_{3-j}|^2) \\ & + 2B(|u_1|^2 + |u_2|^2)] \\ & + (1 + B)u_j u_{3-j}^* v_{3-j}; \quad j = 1, 2; \end{aligned} \quad (1.7)$$

where a prime denotes derivative with respect to z and

$$I_\Sigma = |u_1|^2 + |u_2|^2 + |v_1|^2 + |v_2|^2 = I_1 + I_2. \quad (1.8)$$

In (1.6) and (1.7) we have already taken into consideration that $A + B = 1$. Then, as a next step, since variables $u_j(z)$ and $v_j(z)$ are complex, we introduce their real amplitudes [respectively $p_j(z)$ and $q_j(z)$] and phases, $\phi_j(z)$ and $\psi_j(z)$:

$$u_j = p_j e^{i\phi_j}; \quad v_j = q_j e^{i\psi_j}; \quad j = 1, 2. \quad (1.9)$$

Substituting now (1.9) into (1.6) and (1.7) and introducing a new, compound, phase η :

$$\eta = \phi_1 - \phi_2 - \psi_1 + \psi_2 \quad (1.10)$$

one gets the equations governing the spatial dynamics of the amplitudes and phases:

$$\begin{aligned} (p_1^2)' = -(q_2^2)' = (p_2^2)' = -(q_1^2)' \\ = \chi k(1 + B)p_1 p_2 q_1 q_2 \cdot \sin \eta \end{aligned} \quad (1.11)$$

$$\begin{aligned} -2(-1)^j \cdot (\chi k)^{-1} p_j^2 \phi_j' = & p_j^2 [(1 - B)I_\Sigma + 2B(q_1^2 + q_2^2)] \\ & + p_1 p_2 [(1 - B)p_1 p_2 \\ & + (1 + B)q_1 q_2 \cos \eta]; \quad j = 1, 2 \end{aligned} \quad (1.12)$$

$$\begin{aligned} -2(-1)^j (\chi k)^{-1} q_j^2 \psi_j' = & q_j^2 [(1 - B)I_\Sigma \\ & + 2B(p_1^2 + p_2^2)] \\ & + q_1 q_2 [(1 - B)q_1 q_2 \\ & + (1 + B)p_1 p_2 \cos \eta]; \\ & j = 1, 2 \end{aligned} \quad (1.13)$$

i.e., eight equations all together. The number of these

equations can further be reduced to five by combining four equations (1.12) and (1.13) into one for the phase η :

$$2(\chi k)^{-1} \eta' = (1 - 5B)(p_1^2 + p_2^2 - q_1^2 - q_2^2) + (1 + B) \cdot (p_1^{-2} + p_2^{-2} - q_1^{-2} - q_2^{-2}) p_1 p_2 q_1 q_2 \cos \eta \tag{1.14}$$

Four independent invariants [18] of those five equations [(1.11) and (1.14)] can readily be found, and therefore, this system of equations can be reduced to one first-order differential equation for only one variable. From (1.11) one immediately gets three first integrals that correspond to three independent invariants (I_1, I_2, J):

$$p_1^2 + q_1^2 = \text{inv} = I_1; \quad p_2^2 + q_2^2 = \text{inv} = I_2 \tag{1.15}$$

$$p_1^2 + q_2 = \text{inv} = J \tag{1.16}$$

which results also in other conservation invariants that are linear combinations of (1.15) and (1.16):

$$p_1^2 + p_2^2 + q_1^2 + q_2^2 = \text{inv} (= I_1 + I_2 = I_\Sigma) \tag{1.17}$$

$$q_1^2 - q_2^2 = \text{inv} (= I_1 - J); \quad p_1^2 - p_2^2 = \text{inv} (= J - I_2) \tag{1.18}$$

$$p_2^2 + q_1^2 = \text{inv} (= I_\Sigma - J). \tag{1.19}$$

The invariants (1.15) imply conservation of the energy of both of the left and right traveling waves separately [as well as conservation of the energy of the total field (1.17)], whereas, once expressed in terms of the circularly polarized components, the equation (1.16) [as well as (1.18) and (1.19)] obviously imply conservation of the circular momenta of the two waves [18]. Using now (1.11) and (1.14) one finally gets the fourth invariant $M(1-5B)$ related to the interference between the two counterpropagating waves:

$$2(1 + B) p_1 p_2 q_1 q_2 \cos \eta + (1 - 5B) \cdot (p_1^2 p_2^2 + q_1^2 q_2^2) = \text{inv} = M(1 - 5B). \tag{1.20}$$

This invariant, along with (1.16) and (1.19), constitutes the conservation of total field momentum in a Kerr-nonlinear medium [18], [21]. Now, using four independent invariants I_1, I_2, J , and M [(1.15), (1.16), and (1.20)] and introducing the intensities of all the circular components

$$P_j = p_j^2; \quad Q_j = q_j^2; \quad j = 1, 2 \tag{1.21}$$

one can reduce the initial set of eight equations (1.11)–(1.13) to one equation for one of the variables, e.g., P_1 :

$$2(\chi k)^{-1} P_1' = \sqrt{4(1 + B)^2 P_1 P_2 Q_1 Q_2 - (1 - 5B)^2 (P_1 P_2 + Q_1 Q_2 - M)^2} \tag{1.22}$$

where

$$P_2 = P_1 + I_2 - J; \quad Q_1 = I_1 - P_1; \quad Q_2 = J - P_1 \tag{1.23}$$

and constants I_1, I_2, J , and M are to be found from the boundary conditions (1.9), which may be now more appropriately written in the form

$$P_1 = P_{10}; \quad Q_1 = Q_{10}; \quad \phi_1 = \phi_{10}; \tag{1.24}$$

$$\psi_1 = \psi_{10} \text{ at } z = 0$$

$$P_2 = P_{2L}; \quad Q_2 = Q_{2L}; \quad \phi_2 = \phi_{2L};$$

$$\psi_2 = \psi_{2L} \text{ at } z = L \tag{1.25}$$

where all eight boundary quantities are readily found from (1.4), if the definitions (1.5), (1.9), and (1.21) are taken into consideration. Therefore, the solution to (1.22) at $z = L$ is

$$\frac{\chi k L}{2} = \int_{P_{10}}^{P_1(L)} \frac{d\xi}{\sqrt{\Phi(\xi)}} \tag{1.26}$$

where

$$\begin{aligned} \Phi(\xi) = & 4(1 + B)^2 \xi (\xi + I_2 - J) (I_1 - \xi) (J - \xi) \\ & - (1 - 5B)^2 [(\xi + I_2 - J) \xi \\ & + (I_1 - \xi) (J - \xi) - M]^2 \end{aligned} \tag{1.27}$$

The respective total intensities I_1 and I_2 are found immediately

$$I_1 = P_{10} + Q_{10}; \quad I_2 = P_{2L} + Q_{2L} \tag{1.28}$$

whereas the quantities $P_1(L), J$, and M are still to be found from the boundary conditions. In general case, the integral in (1.26) can be expressed through the elliptic integrals; however in the particular cases $B = \frac{1}{2}$ and $B = 0$, it can be readily solved analytically via trigonometric functions.

B. Exact Solutions ($B = 0; B = \frac{1}{2}$)

The existence of an analytical solution in the cases $B = 0$ and $B = \frac{1}{2}$ is of considerable importance not only because it provides new insights in the theory of four-wave mixing bistability (and four-wave interactions in general), but also because there is an entire class of materials that fall into one of these particular cases. Indeed, although the Kerr-nonlinearity with $B = 0.5$ should apparently be attributed to more than one particular physical mechanism, the case with $B = 0$ corresponds to the nonlinear susceptibility resulting from the electrostriction mechanism [14], [15] which is peculiar to the broad class of materials. Moreover, one may even find it curious to notice that not withstanding the fact that the mechanisms with $B = 0$ show no light-induced anisotropy for the single-

wave propagation (e.g., they do not produce any light-induced rotation of polarization when the single incident laser beam is elliptically polarized unlike any other mechanisms with $B \neq 0$, see [12] and [13]), those mechanisms (with $B = 0$) are capable to demonstrate such a strong anisotropic nonlinear effect as optical multistability with two counterpropagating waves.

For these two particular cases, the integral in (1.26) is dramatically simplified. This is because the function $\Phi(\xi)$, (1.27), which in general case is a fourth order polynomial, reduces to a second order polynomial when either $B = \frac{1}{2}$ or $B = 0$. Indeed, in such a case it is written as

$$\Phi(\xi) = \alpha^2(B)(I_\Sigma^2 - 4M)[P_M^2 - (\xi - \bar{P}_1)^2] \quad (2.1)$$

where $\alpha = 1 + B$, i.e.,

$$\alpha(B = 0) = 1 \quad \text{and} \quad \alpha(B = \frac{1}{2}) = \frac{3}{2} \quad (2.2)$$

and

$$P_M^2 = 4M(I_1 I_2 - M) \cdot [J(I_\Sigma - J) - M]/(I_\Sigma^2 - 4M)^2 \quad (2.3)$$

$$\bar{P}_1 = \frac{1}{2}[J + (I_2 - I_1) \cdot (2M - JI_\Sigma)/(I_\Sigma^2 - 4M)] \quad (2.4)$$

with

$$J = P_1(L) + Q_{2L}, \quad (2.5)$$

and the explicit solution for the output intensity $P_1(L)$ is readily found as a function of the incident intensity P_{10} and yet unknown constants M and J

$$P_1(L) = \bar{P}_1 - \sqrt{P_M^2 - (P_{10} - \bar{P}_1)^2} \sin\left(\frac{\alpha}{2} \chi L k \sqrt{I_\Sigma^2 - 4M}\right) + (P_{10} - \bar{P}_1) \cos\left(\frac{\alpha}{2} \chi L k \sqrt{I_\Sigma^2 - 4M}\right). \quad (2.6)$$

In the particular case when one of the beam (e.g., the second one) is circularly polarized at the plane of incidence, e.g.,

$$Q_{2L} = 0; \quad P_{2L} = I_2 \quad (2.7)$$

one has [see also (1.20)]

$$J = P_1(L); \quad M = P_1(L) I_2. \quad (2.8)$$

Therefore by introducing dimensionless notations

$$\frac{P_{10}}{I_1} = \rho_0; \quad \frac{P_1(L)}{I_1} = \rho; \quad \frac{1}{2} \alpha \chi k L I_j = N_j \quad (j = 1, 2) \quad (2.9)$$

and substituting (2.7)–(2.9) into (2.6), one gets the relationship determining the normalized output intensity of one of the circular components, ρ , as a function of its incident normalized intensity ρ_0 , and normalized total intensities of both of the waves N_2 and N_1 :

$$\rho_0 = \rho \left[1 - \frac{4N_1 N_2 (1 - \rho)}{N_\Sigma^2 - 4N_1 N_2 \rho} \sin^2 \left(\frac{\sqrt{N_\Sigma^2 - 4N_1 N_2 \rho}}{2} \right) \right] \quad (2.10)$$

where $N_\Sigma = N_1 + N_2$. In the case when both of the waves have the same total intensity, $I_1 = I_2 = I$ (or $N_1 = N_2 = N$), one gets an even simpler equation:

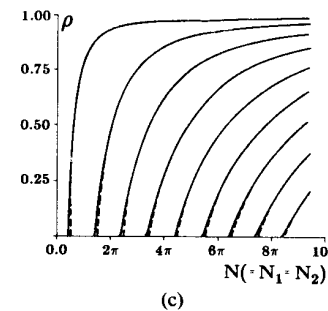
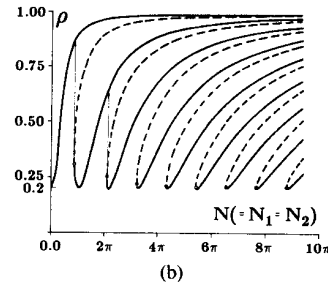
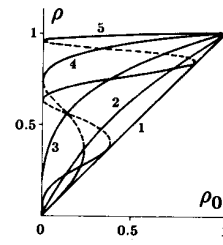


Fig. 1. The normalized output intensity, ρ , of one of the circular components (in the case $B = 0$, and $B = 0.5$) (a) versus the normalized input intensity of the same component ρ_0 when the total intensities of both of the counterpropagating waves are fixed and equal to each other ($N_1 = N_2 = N$), (curves: 1— $N = 0$, 2— $N = \pi/4$, 3— $N = \pi/2$, 4— $N = \pi$, 5— $N = 5\pi/2$); (b) and (c) versus N when ρ_0 is fixed; $\rho_0 = 0.2$ (b), and $\rho_0 = 10^{-5}$ (c). The presumable stable branches of the curves are depicted by solid lines, whereas the presumable unstable branches are shown by dash lines. The transition from one of the stable branches to another occurs by jump. The first two jumps are shown at Fig. 1(b) by arrows. Since the locations of the remaining jumps are obviously similar to the first two jumps, they are indicated only by thick circles.

$$\rho_0 = \rho \cos^2(N\sqrt{1 - \rho}). \quad (2.11)$$

For the latter case ($N_1 = N_2 = N$), the behavior of the normalized output intensity, ρ , of one of the circular components as a function of the respective normalized input intensity ρ_0 for the fixed total intensities of both of the waves N is depicted at Fig. 1(a), and as a function of N for the fixed ρ at Fig. 1(b). From (2.11), one can see that in the case of varied ρ_0 , bistability appears when $\pi/2 < N < 3\pi/2$, and in general case, multistability of n th order appears when

$$(n - \frac{3}{2})\pi < N < (n - \frac{1}{2})\pi \quad (2.12)$$

where n is integer ≥ 2 [see Fig. 1(a)]. In the case of varying N [Fig. 1(b)] with fixed ρ_0 , the multistability can be attained for any ρ_0 , although it is much easier when ρ_0 is small. If $\rho_0 \ll 1$, the points of “off-jumps” are determined by $N = \pi(n - \frac{3}{2})$, i.e., the minimal required intensity is still

$$N_{cr} = \frac{(1+B)}{2} \chi k L I_{cr} = \pi/2. \quad (2.13)$$

Since the case of $\rho_0 = 0$ [and therefore $P_{10} = 0$, see (2.9)] and $Q_{2L} = 0$ corresponds to one of the four possible nonlinear eigenmodes [18], specifically that one with two (counterpropagating) *counterrotating* circularly polarized waves, one can readily see that this particular eigenmode becomes *spatially unstable* when $N > N_{cr}$. Particularly, the behavior for very small ρ_0 ($\rho_0 = 10^{-5}$) is shown in Fig. 1(c). Contrarily, the eigenmode with *corotating* circularly polarized waves (e.g., $Q_{10} = Q_{2L} = 0$, i.e., $\rho_0 = 1$), corresponds to the trivial solution

$$\rho = \text{const} = \rho_0 = 1 \quad (2.14)$$

and is *spatially stable*. The actual investigation of the stability of the system in consideration (including the stability of isolas) should be based on the dynamic equation of the nonlinear system which must include time dependent propagation terms. For example, the dynamics of a weak probe signal in the presence of a strong pump in four-wave mixing was considered in [22]. We plan to address the stability of isolas with strong waves in the future. In general case $0 < \rho_0 < 1$, the interesting feature of the curves at Fig. 1(b) and (c) is that although the system demonstrates multistability, it does *not* show any hysteresis in the following sense. Suppose that the intensity N increases (sufficiently slow) from zero; it can be readily seen then that the system will be bounded within only *one* branch (specifically that one which starts from the point $N_1 = 0$, $\rho = \rho_0$) and will never jump to any other branch. In order to attain any jumps, one has to somehow settle the system at any higher order branches [with N sufficiently high, $N > N_{cr}(\rho_0, n)$], e.g., with transient process (which is not discussed in this paper) and make N to slowly decrease, which will result in several consecutive jumps to lower order branches (but not vice versa). However, if one again increases the intensity N at any branch, the system will be bound to only that particular branch. This situation (i.e., existence of only “off-jumps,” and no “on-jumps”) is quite a distinct feature of four-wave multistability which is not seen in any other kind of optical bistability. It is not quite clear yet whether this feature is due to any idealization in the formulation of the problem (e.g., due to the fact that no dissipation was taken into consideration) or it is an intrinsic property of the entire process.

The most interesting manifestation of apparently the same property is still another distinct feature of the process which reveals itself in the appearance of multiple limited isolated branches of the solution when one of the intensities (e.g., N_2) is fixed while another one (N_1) varies. These isolated branches (which are called isolas) are shown in Fig. 2 for various ρ_0 ; they appear only for $N_2 > N_{2cr}(\rho_0) > 0$. If $\rho_0 \ll 1$ [see e.g., Fig. 2(b) and (c)], all of the isolas are located in the vicinity of $N_1 = N_2 = N$, and their total number s is estimated as

$$s = \left[\frac{N}{\pi} + \frac{1}{2} \right], \quad (\text{with } s \geq 1) \quad (2.15)$$

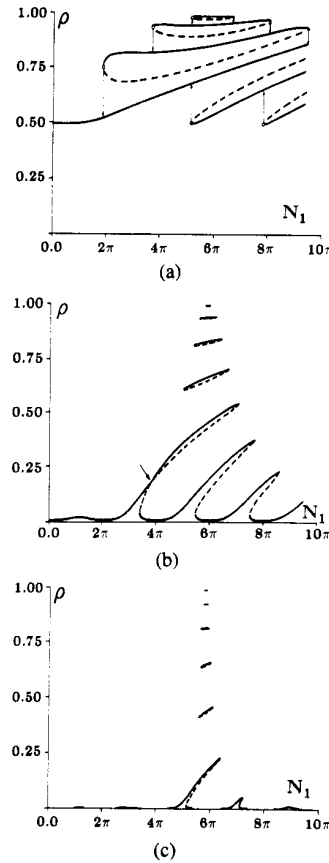


Fig. 2. A plot of ρ versus the total intensity of one of the fields, N_1 , when the other one is fixed ($N_2 = 6\pi$) for different ρ_0 in the cases $B = 0$ and $B = 0.5$: (a) $\rho_0 = 0.5$ (this corresponds to a linear polarization of the wave E_1 at the plane of incidence); (b) $\rho_0 = 0.01$ (the point of the formation of a new isola is shown by a short arrow); (c) $\rho_0 = 5 \times 10^{-4}$. The jumps are shown by arrows only at Fig. 2(a).

where square brackets denote the integer part of the argument. The “altitude” ρ_n of the n th isola is estimated then as

$$\rho_n \sim 1 - \pi^2(n + \frac{1}{2})^2/N^2 > 0 \quad (2.16)$$

(with $N > N_{cr} = \pi/2$). One may see the formation of the isolated isola at Fig. 2(b), where the point of the waist between a “new-born” isola and the “mother” curve is shown by a short arrow. In Fig. 2(c), one isola is still missing as compared to the estimation in (2.15). The sixth isola will appear when ρ_0 tends even further to zero. The whole entity of solution branches (regardless of existence of isolated branches) is enveloped by two simple curves: from below by

$$\rho = \rho_0 \quad (2.17)$$

(i.e., there is no solution with $\rho < \rho_0$), and from above, by

$$\rho_{\text{env}} = \frac{\rho_0}{1 - 4N_1N_2(1 - \rho_0)/N_{\Sigma}^2}, \quad (\text{but } \rho_{\text{env}} < 1) \quad (2.18)$$

e.g., for $\rho_0 \ll 1$

$$\rho_{\text{env}} \approx \rho_0 \left(\frac{N_1 + N_2}{N_1 - N_2} \right)^2, \quad (\text{but } \rho_{\text{env}} < 1). \quad (2.19)$$

Due to the fact that the isolas are isolated curves, they again feature nonhysteretic multistability: only "off-jumps" can be observed with regard to these branches with no "on-jump." However, at the same time the total solution still features a conventional hysteretic multistability (see Fig. 2) which was first demonstrated in [1], [8].

Let us estimate the critical pumping intensity, I_{cr} , required to obtain one single isola. By choosing a typical electrostrictive medium (e.g., CCl_4) and applying a sufficiently long pulse, we make electrostriction prevail which corresponds to the case $B = 0$. Under such conditions, the critical intensity for one isola is given by (2.13) and (2.15)

$$I_{cr} = \frac{1}{4K_\rho L} \quad (2.20)$$

where K_ρ is the electrostrictive coefficient [14]. For CCl_4 , $K_\rho = 1.21 \times 10^{-7}$ esu [14] and the nonlinear coefficient for the refractive index, n_2 is given by $n_2 = 1/2 K_\rho \lambda$ where λ is the wavelength. Therefore, for $L = 10$ cm one obtains the critical electric field, $E_{cr} = 1.36 \times 10^6$ V/cm.

C. Numerical Solutions ($B = \frac{1}{5}$)

The isolated branches of multistable solution is a feature which is peculiar not only to the particular cases $B = 0$ and $B = \frac{1}{2}$, when it could be demonstrated by analytical method, but also to any other values of B . In order to demonstrate this, we consider a case in which $B = \frac{1}{5}$. In this case a right-side part of (1.26) is in the form of elliptic integral; now one can readily simplify (1.26) into

$$1.2 \chi k L = \int_{P_{10}}^{P_1} \frac{dP_1}{\sqrt{P_1 P_2 Q_1 Q_2 - \bar{M}^2}} \quad (3.1)$$

where P_2 , Q_1 , and Q_2 are related to P_1 by (1.23) and \bar{M} is some constant to be found from the boundary conditions. Similarly to Section B, we further simplify (3.1) with the condition that one of the waves (e.g., E_2) has either left or right circular polarization at the plane of incidence, which along with condition (2.7) yields [see also (1.20)]

$$J = P_1(L), \quad \text{and} \quad \bar{M} = 0 \quad (3.2)$$

Expressing (3.1) in terms of dimensionless variables

$$\rho_0 = \frac{P_{10}}{I_1}; \quad \rho = \frac{P_1(L)}{I_1};$$

$$\zeta = \frac{P_1}{I_1}; \quad N_j = 1.2 \chi k I_j L \quad (j = 1, 2) \quad (3.3)$$

(note that N_j differ now from the dimensionless intensities N_j (2.8) by the numerical factor), and taking into consideration the boundary conditions (2.7) and (3.2) one gets the equation for the output intensity ρ of one of the circular components

$$N_1 = \int_{\rho_0}^{\rho} \frac{d\zeta}{\sqrt{\zeta(1-\zeta)(\rho-\zeta)(\zeta-\rho+N_2/N_1)}} \quad (3.4)$$

which can be expressed in terms of elliptic integrals [23]

$$\frac{1}{2} \sqrt{N_1 N_2} = nK \left(\sqrt{S \frac{N_1}{N_2}} \right) + F \left(\arcsin \sqrt{\frac{\rho - \rho_0}{\rho(1 - \rho_0)}}, \sqrt{S \frac{N_1}{N_2}} \right) \quad (3.5a)$$

for

$$1 > \rho > \rho_0 \geq 0 > \rho - N_2/N_1$$

or

$$\frac{\sqrt{S}}{2} N_1 = nK \left(\sqrt{\frac{1}{S} \cdot \frac{N_2}{N_1}} \right) + F \left(\arcsin \sqrt{S \frac{N_1}{N_2} \frac{\rho - \rho_0}{\rho(1 - \rho_0)}}, \sqrt{\frac{1}{S} \frac{N_2}{N_1}} \right) \quad (3.5b)$$

for

$$1 > \rho > \rho_0 \geq \rho - N_2/N_1 > 0$$

where

$$S = \rho(1 - \rho + N_2/N_1)$$

n is an integer; $K(x)$ and $F(y, x)$ are complete and incomplete elliptic integrals of the first kind which are determined as

$$K(x) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - x^2 \sin^2 \theta}} \quad (3.6)$$

and

$$F(y, x) = \int_0^y \frac{d\theta}{\sqrt{1 - x^2 \sin^2 \theta}} \quad (3.7)$$

respectively.

The behavior of the normalized output intensity, ρ , is plotted against input intensity, ρ_0 , in Fig. 3(a) and against total intensity, N_1 , in Fig. 3(b) and (c) when $N_1 = N_2 = N$, and against N_1 in Fig. 4 when N_2 is fixed. The curves in Fig. 3 are similar to those in Fig. 1 for exact solution ($B = 0$ or 0.5). One may see that when the total intensity, N_1 is greater than some critical value, $N_{cr} = \pi$, multistability appears. Similar to Section B, one can see again that $\rho_0 = 0$ (counterrotating circularly polarized waves) and $\rho_0 = 1$ (corotating circularly polarized waves) correspond to eigenmodes, with the latter one always stable, and the former one unstable when $N > N_{cr}$. The behavior of ρ for ρ_0 close to zero ($\rho_0 = 10^{-4}$) is illustrated in Fig. 3(c). Considering Fig. 3(b), one can see that the minimal output intensity, ρ , varies periodically with respect to total intensity N [similar to that one at Fig. 1(b)], i.e., is equal to the input intensity, ρ_0 , after a fixed interval. Portions

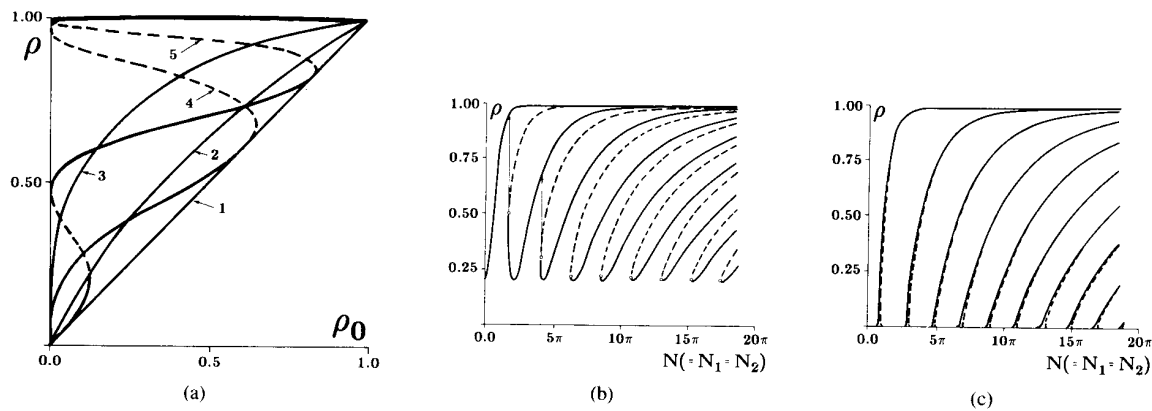


Fig. 3. The same as Fig. 1, in the case $B = 0.2$, (a) curves: 1— $N = 0$, 2— $N = \pi/2$, 3— $N = \pi$, 4— $N = 3\pi$, 5— $N = 4\pi$, (b) $\rho = 0.2$, (c) $\rho_0 = 10^{-4}$.

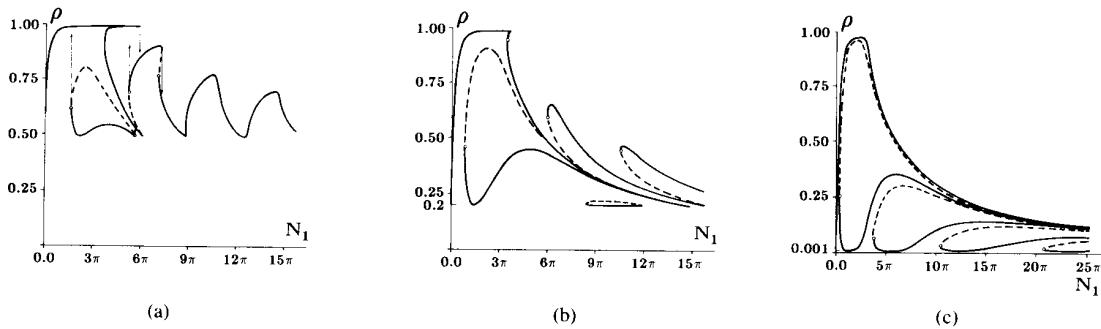


Fig. 4. The same as Fig. 2, in the case $B = 0.2$, $N_1 = 3\pi$; (a) $\rho_0 = 0.5$ (E_1 is linearly polarized at the plane of incidence), (b) $\rho_0 = 0.2$, (c) $\rho_0 = 10^{-3}$.

of curves in Fig. 3(b) around $\rho = 1$ extend to infinity and run almost parallel to each other, similar to Fig. 1(b). This results again in nonhysteretic behavior common to both Figs. 1 and 3. The lower branch of the curve can jump to the upper branch but not vice versa. One may somehow excite the system to higher order branches with sufficiently large total intensity and keep the system in that branch until the total intensity decreases down to some critical value characteristic for that particular branch that causes the system to jump to lower order branch (see Section B). Moreover, another interesting feature, isolated branches (isolas) that is similar to those of Fig. 2 can be explored in Fig. 4. As these isolated branches go further away from "the mother curve" which is the longest branch, their sizes decrease gradually. The curve which is forming an envelope for the entire entity of curves at Fig. 4, can be expressed by the relationships: $\rho = \rho_0$, from the below [the same as (2.17)], and by

$$\rho = \frac{N_2}{N_1} + \rho_0 \quad \text{for } N_1 > \frac{N_2}{1 - \rho_0}. \quad (3.8)$$

from the above. Contrary to the case with $B = 0$ and $B = \frac{1}{2}$ (Fig. 2), the isolas appear now in the lower portion of Fig. 4 with decreasing sizes as they go downward. In the far right end of the curves $\rho(N_1)$, i.e., when N_1 gets sufficiently large, the multistability disappears and only vanishing oscillation remains which could be approxi-

mately described by the following expression which follows from (3.4) with the assumption that $\rho - \rho_0 \ll 1$:

$$\rho = \rho_0 + \frac{N_2}{N_1} \sin^2 \left[\frac{N_1}{2} \sqrt{\rho_0(1 - \rho_0)} \right]. \quad (3.9)$$

Comparing Fig. 4(a) and (b), one can see that the output intensity curves change as the initial intensity, ρ_0 , varies. At $\rho_0 = 0.5$ [Fig. 4(a)], there is only one isolated branch, whereas for $\rho_0 = 0.2$ [Fig. 4(b)] the formation of another (smaller) isola occurs. When ρ_0 is decreasing, the number of isolas increases; this tendency is manifested in Fig. 4(c). Further, (3.5) can be simplified if $\rho_0 = 0$.

$$\frac{1}{2} \sqrt{N_1 N_2} = nK \left(S \sqrt{\frac{N_1}{N_2}} \right). \quad (3.10)$$

One can see the number of solutions for ρ increases with N_1 and N_2 .

CONCLUSION

We found that a four-wave mixing optical bistability in addition to "conventional" hysteretic multistable behavior can demonstrate a remarkable feature: multiple isolated nonhysteretic branches of solution. These branches are first found for the tensors of third-order nonlinear susceptibility which allow for an analytical solution ($B = 0$, and $B = 0.5$), one of which ($B = 0$) is also of practical importance since it corresponds to the electrostriction mech-

anism of optical nonlinearity. The numerical calculations for other third-order nonlinearities (in particular, $B = 0.2$) show that this feature is quite a universal one. One may assume that this phenomenon is due to the formation of light-induced "internal" spatial resonances in a steady-state regime of excitation. These resonances can become a very important component in the theory of four-wave mixing instability, self-oscillations and chaos. Silberberg and Bar-Joseph showed [24] that even for (counterpropagating) waves which are linearly polarized in the same plane (this particular polarization configuration does not provide a steady-state bistability considered in this paper), dramatic instabilities in time domain and even a chaos may occur, if the mechanism of nonlinearity has an appropriate relaxation time. Those instabilities have distinct resonant frequencies in a time domain. This suggests that in the situation when these frequencies come to resonance with the frequencies pertinent to the spatial nonlinear resonances which are due to isolated states discussed in this paper, one may expect very rich behavior of the system both in time and space domains. We are planning to address some of these effects in the near future.

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