Multistable Self-Trapping of Light and Multistable Soliton Pulse Propagation

A. E. KAPLAN

Abstract—**It is demonstrated that a nonlinear Schrödinger equation with certain nonlinearities allows for an existence of multistate single solitons (i.e., single solitons with the same carried power but different propagation parameters). In nonlinear optics, these solitons may exist either in the form of short bistable pulses, or bistable self-trapping (both two- and three-dimensional).**

I. Introduction

In this paper, we consider two closely related (from the mathematical standpoint) problems relevant to the nonlinear light propagation:

1) CW self-trapping of light in infinite (or semi-infinite) media with a nonlinearity which allows for the formation of a spatially steady self-channel [1]–[4], and

2) pulse propagation of a plane wave envelope in a medium [4], [5] (or of a confined single mode in an optical fiber waveguide [6]) with an appropriate nonlinearity and frequency dispersion which allows for the formation of a single nondispersive pulse (soliton).

Both propagation situations are governed by the same kind of a wave equation, the so-called nonlinear Schrödinger equation [7]. Under the appropriate conditions, this equation may have a singular nondispersive solution (a single soliton), which corresponds to either a self-channel [in the case 1)] or a single pulse [in the case 2)]. One of the main properties of these single solitons of the nonlinear Schrödinger equations studied to data is that their propagation characteristics (such as the propagation constant of the channel or the speed of the pulse, the transversal size of the channel or the time duration of the pulse) are *single-valued* functions of the total power carried by the soliton (either the channel or the pulse). This holds particularly true for Kerr-like nonlinear media.

The purpose of this paper is to demonstrate that for a certain class of nonlinearities, the propagation characteristics of a single soliton become *multivalued*. This implies that there is more than one possible propagation constant (and respectively more than one possible size of the intensity profile) of the single soliton for the same total power carried by it. We shall call such a soliton a multistate soliton (or multistable soliton if more than one of its steady states is stable).

We show that the existence of the multistate solitons of

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The author is with the School of Electrical Engineering, Purdue University, W. Lafayette, IN 47907.

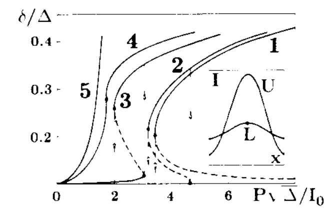

Fig. 1. A propagation constant δ versus the total power P carried by the soliton. Curves 1–5 corresponds to various functions of nonlinearity: 1—step-function (13); 2—$f = a_2 I^2 + a_3 I^3 - a_4 I^4$, ($a_2, a_3, a_4 > 0$); 3—(18) with $a_1 a_3 < a_2^2 S_{cr}$; 4—(18) with $a_1 a_3 = a_2^2 S_{cr}$; 5—Kerr-nonlinearity, $f \propto I$. The broken lines at curves 1–3 correspond to the unstable solitons. In the insertion, the intensity profiles $I(x)$ are depicted of solitons that carry the same power but correspond to different branches of function $\delta(P)$—upper branch (U) and lower branch (L).

the (generalized) nonlinear Schrödinger equation (Section II) is related to the type of dependence of the nonlinear susceptibility on the intensity of light. For example, the multistate soliton waves cannot be observed in a Kerr-like nonlinear medium; they may exist only if the nonlinear component of the susceptibility as function of intensity is either changing its sign or if its derivative has a sufficiently sharp peak (e.g., is a step-like function [8]). This may be, e.g., due to such mechanisms as either a light-induced phase transition in a material or multiphoton saturation of resonant levels (see Section V below).

The existence of the multistate solitons as a new property of the nonlinear Schrödinger equation is of considerable interest to soliton theory in general and may find applications in various fields. It would be most natural, though, to envision these solitons as a novel, very interesting optical bistable or switching effect. This may be illustrated using an example in which a laser beam propagates in the nonlinear medium (capable of supporting multistable self-trapping), and the total power of the beam is the lowest required for self-trapping. If the incident power is now slowly increased (for the sake of simplicity we assume also that the intensity profile of the incident beam is constantly adjusted to maintain a spatially steady-state self-channel), at some critical power the beam loses its stability (see, e.g., Fig. 1, curve 3). As a result, its propagation constant jumps to the upper branch (and its size drastically decreases, see insertion in Fig. 1). If after that the incident power is decreased, the reverse jump will happen at the lower power. This results in a hysteresis. For some kind of nonlinearities (see Sections III and V below), the first jump may occur from nontrapped propagation to the self-trapped phenomenon. The switching of the soliton pulse propagation in a fiber waveguide may be performed in the same fashion.

It must be noted, though, that the existence of true hysteresis and optical bistability based on multistate solitons is yet an open question and therefore is to be a subject of further detailed research. The reason for this is that the transformation from one self-channel to another one (or from one soliton pulse to another one) after the jump occurs involves spatial as well as temporal dynamics, whereas the results obtained in this paper are concerned with the steady states only. During the spatial (or temporal) transient regimes some portion of the total power of the initial soliton will be lost in the form of the side radiation (nonsoliton background wave) emitted into infinity. (There is also a possibility of the formation of yet other, additional solitons.) The same uncertainty holds true also for multistate modes in nonlinear waveguides (see the discussion below).

On the one hand, these losses provide a source of energy dissipation which is a necessary condition to attain a true hysteresis in any other system with intrinsic bistability [9]. On the other hand, these losses should be accounted for in order to make sure that the power retained by the surviving soliton (if single) is sufficient to keep it at the upper branch of the hysteretic curve. Although the explicit answer to this question is still to be found, it is obvious that regardless of the hysteresis, a very strong switching effect may be observed at the output of the system (e.g., a change in the shape of the pulse at the end of nonlinear fiber waveguide or in the intensity profile of the self-trapped light transmitted through the exit wall of the thick nonlinear layer). One may note also that due to the fact the nonlinearities required to attain a multistate solitons must have a saturation or be otherwise limited, no self-collapse effect is expected. The preliminary consideration (see Section IV below) shows also that the solitons that belong to the upper and lower branches of the hysteretic curve, are stable.

The optical bistable or switching device may become one of the simplest from the electrodynamic standpoint. Indeed, it does not use any resonator [10], a single nonlinear interface [11], [12], nonlinear waveguides [13], [14] formed by two nonlinear interfaces (i.e., a sandwich structure with a linear layer between two semi-infinite nonlinear layers [13], [14] or vice versa [13]), an intrinsically bistable media [8], or any other electrodynamic component except a nonlinear layer or nonlinear optical fiber with the wave propagating in only one direction. One may note, though, that the existence of multiself-trapping in the interfaceless nonlinear medium with special nonlinearity somewhat resembles existence of multistate modes in the two-interface nonlinear waveguides [13], [14] with Kerr-like nonlinearity. This emphasized that the same problem discussed above with respect to multistate solitons holds true with respect to modes in the nonlinear waveguides. Namely, all the theoretical calculations done so far for the nonlinear waveguides [14], revealed in fact only the existence of multistate *steady* modes, which does not warrant at all observable bistability of "input-output" characteristics of the transmitted wave. During the switching, the modes should undergo the spatial transition which leaves a still unresolved uncertainty as to the existence of true hysteresis and bistability. Nevertheless, whether multivalued steady state solutions result in either true bistability or sharp switching, the systems discussed here are attractive since they do not have any nonlinear interfaces along the direction of propagation (in contrast to nonlinear interface effects [11]–[14]); therefore they do not require a very close matching [11]–[14] of refractive indexes of all layers in the nonlinear waveguides.

With regard to the multistate pulse propagation in nonlinear optical fibers, there is a reasonable hope that it may provide the first (to the best of our knowledge) known opportunity to attain a temporal (or dynamic) bistability. This is in contrast to all known kinds of optical bistability which were so far formulated in terms of temporally steady-state regimes. The very notion of temporally steady-state optical bistability comes into the inevitable contradiction with the applications, most of which assume fast pulse regime of operations. When exploited in a dynamic regime, such effects still demonstrate hysteretic behavior which, however, can hardly be identified with the original "adiabatic," steady-state hysteresis. The dynamic hysteresis is affected more strongly by the relaxation processes than by steady-state bistable states, especially when the total switching cycle has the duration time of the same order as relaxation times. The truly dynamic (or temporal) bistability which offers a potential to resolve this contradiction may be based on multivalued duration of the soliton pulses discussed in this paper.

Both of these effects (multistate self-trapping of light and multistate solition pulses) may be viewed as the ultimate manifestation of the multistable wave propagation, since they are based on the simplest possible electrodynamic configuration. They may also provide new opportunities in the field of optical bistability.

II. Wave Equation

We assume that the EM field $\vec{E}(\vec{r})\, e^{i(\vec{k}\vec{r}-\omega t)}$ propagates in a lossless medium having intensity-dependent susceptibility ϵ, such that

$$\epsilon = \epsilon^L + \epsilon^{NL}(|E|^2) \tag{1}$$

and introduce $f(|E|^2) = \epsilon^{NL}/\epsilon^L$ where f can be an arbitrary function of the intensity $|E|^2$ with $f(0) = 0$. The case of $f \propto |E|^2$ corresponds to the Kerr-nonlinearity. We assume also that the axis of the propagation is $\tilde{z}$ (i.e., $\vec{k} = k\,\hat{e}_{\tilde{z}}$, where $k^2 = \omega^2\epsilon^L/c^2$) and introduce dimensionless coordinates $z = k\tilde{z}$, $x = k\tilde{x}$, $y = k\tilde{y}$. Then, from the Maxwell's equations, in the conventional approximation of a slowly varying envelope [2], [3] ($\partial^2E/\partial z^2 << \Delta_\perp E$, where $\Delta_\perp$ is a Laplacian operator in a plane normal to the z axis), one readily gets the (generalized) nonlinear Schrödinger equation governing the nonlinear wave propagation:

$$2i\partial E/\partial z + \Delta_\perp E + Ef(|E|^2) = 0. \tag{2}$$

In the two-dimensional case (e.g., with $\partial/\partial y = 0$), this equation is reduced to the form

$$2i\partial E/\partial z + \partial^2 E/\partial x^2 + Ef(|E|^2) = 0. \quad (3)$$

In the case of one-dimensional pulse propagation along the z_1 axis in a slightly dispersive medium with nonlinearity $f_1(|E|^2)$, the equation of the propagation is [3], [6]

$$2i\partial E/\partial z_1 + (dv/d\omega)\, v^{-2}\partial^2 E/\partial \xi^2 + kEf_1(|E|^2) = 0 \quad (4)$$

where $\xi = t - z_1/v$; $v = d\omega/dk$ is the group velocity of linear propagation. Equation (4) can readily be transformed into (3) by proper scaling, e.g., by assuming

$$z_1 = z/k^2 \cdot (dv/d\omega); \qquad \xi = x/kv; \quad f_1 = f \cdot k \cdot (dv/d\omega). \quad (5)$$

The slowly varying envelope approximation implies either small diffraction [(2), (3)] or small dispersion [(4)].

The stationary solutions (in particular single solitons) of (2)–(4) have nonvarying intensity profile, $\partial|E|^2/\partial z = 0$, i.e., such solutions are written in the form

$$E(x, y, z) = u(x, y) \exp (i\delta z/2 + i\phi) \quad (6)$$

where $u = |E|$ is a real amplitude of the field, ϕ = count is a real phase and δ is (unknown) real speed (or propagation constant) of the soliton.

III. Bistable Soliton Pulses and Two-Dimensional Self-Trapping

Two-dimensional self-trapping and one-dimensional soliton pulse propagation (e.g., in fibers) are governed by the same equation, (3). Substituting (6) into (3) and bearing in mind that in this case $u = u(x)$, one gets the equation for the real amplitude u

$$d^2u/dx^2 + u|f(u^2) - \delta| = 0 \quad (7)$$

whose soliton solution must satisfy the condition $u \to 0$ as $|x| \to \infty$ in order for the total power $P = \int_{-\infty}^{\infty} u^2\, dx$ to be limited. This provides for the first integral of (7) in the form

$$(du/dx)^2 = 2 \int_0^u u[\delta - f(u^2)]\, du \quad (8)$$

integration of which gives the soliton amplitude profile $u(x)$ for each particular δ and $f(u^2)$. In order to evaluate a total power P, however, one needs not know an explicit form $u(x)$. Indeed, by making use of (8) and introducing $I = u^2 = |E|^2$, one shows that

$$P(\delta) = \int_0^{I_m(\delta)} dI/\sqrt{\delta - F(I)} \quad (9)$$

where

$$F(I) = I^{-1} \int_0^I f(I)\, dI, \qquad (F(0) = 0), \quad (10)$$

i.e., P is determined immediately by $f(I)$ and δ. In (9), $I_m(\delta)$ is a peak intensity of the soliton; it is defined as a minimal positive root of the equation $F(I) = \delta$. The multistability of a single soliton is realized when the function $\delta(P)$ implicitly determined by (9) becomes multivalued.

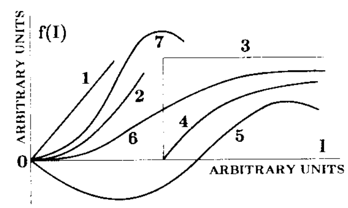

Fig. 2. Various functions of nonlinearity f versus the field intensity I (see details in the text).

It is readily shown that a Kerr-like nonlinearity ($f \propto I$), Fig. 2, curve 1, results only in a one-valued single soliton (with $\delta \propto P^2$), see Fig. 1, curve 5; the same is valid also for any other nonlinearity with $f \propto I^\mu$ where $\mu > 0$ (but $\mu \neq 2$). The nonlinearity $f \propto I^2$, Fig. 2, curve 2, plays a special role in the two-dimensional propagation in the sense that in this case the total energy carried by any single soliton is the same regardless of its spatial profile and propagation constant. Indeed, for $f = I^2/I_0^2$ where I_0 = const, the intensity profile $I(x)$ and propagation constant δ are defined [15] from (8):

$$I(x) = I_m/\cosh (2I_m x/I_0\sqrt{3}); \qquad \delta = I_m^2/3I_0^2 \quad (11)$$

where the maximal intensity of the soliton I_m is an arbitrary constant; the total power is $P = \pi/2I_0$. One may note from (9) that in the general case of arbitrary $f(I)$, a constant δ may be viewed as a first integral ("energy") of some system with a potential $F(I)$; see, e.g., (10). The motion of this system in some p domain can then be described by the equation

$$d^2I/dp^2 + 8d[F(I)]/dI = 0 \quad (12)$$

where if p is interpreted as a "time," $P(\delta)$ is a total "period" of oscillation of the system for any given "energy" of excitation δ. Particularly, one may see that the case $f \propto I^2$ (and therefore, $F \propto I^2$) corresponds to a "linear oscillator," with the "period" of oscillation P independent of its "energy" δ, i.e., $dP/d\delta = 0$ as suggested above.

In order to demonstrate the existence of the countable set of states of the single soliton (with more than one state) we consider first the step-nonlinearity (this problem has first been considered in [8]):

$$f(I) = 0, \text{ if } I < I_0, \text{ and } f = \Delta, \text{ if } I > I_0 \quad (13)$$

Fig. 2, curve 3, where I_0 and Δ are some positive constants. Substituting (13) into (9) one gets

$$P(\delta) = \frac{I_0}{\sqrt{\Delta}} \frac{1}{1-\beta}\left(\frac{1}{\sqrt{\beta}} + \frac{\text{arc sin}\sqrt{\beta}}{\sqrt{1-\beta}}\right); \qquad \beta \equiv \frac{\delta}{\Delta}. \quad (14)$$

The function β versus P determined by (14) is a two-valued function (Fig. 1, curve 1) for any $P > P_{cr1} \approx 3.44 I_0/\sqrt{\Delta}$ with $\beta(P_{cr}) \approx 0.21$. The further example is given by the nonlinearity (Fig. 2, curve 4)

$$f(I) = 0, \text{ if } I < I_0, \text{ and}$$
$$f(I) = \Delta(1 - I_0^2/I^2), \text{ if } I > I_0. \quad (15)$$

In contrast with (13), $f(I)$ is now a continuous function, whereas its derivative df/dI is still discontinuous. The total power (9) now is

$$P = \frac{I_0}{\sqrt{\Delta}} \frac{1}{1-\beta}\left(\frac{1}{\sqrt{\beta}} + \frac{\arccos\sqrt{\beta}}{\sqrt{1-\beta}}\right); \qquad \beta = \frac{\delta}{\Delta} \quad (16)$$

which essentially represents the same kind of behavior as (14), i.e., provides two-valued solution $\beta(P)$ for any $P > P_{cr2} \approx 4.28 I_0/\sqrt{\Delta}$ with $\beta(P_{cr}) \approx 0.26$. In these cases, the nontrivial branches of the function $P(\delta)$ tend to infinity as $\delta \to 0$ and $\delta \to \Delta$ (note that the third, "trivial," branch with $\delta \equiv 0$, P-arbitrary, corresponds to a nontrapped beam with $I_m < I_0$). This suggests a bistability without hysteresis and is due to the fact that nonlinearity $f(I)$ differs from zero only for some finite $I > I_0$. The same kind of soliton bistability is exhibited by the system, if either 1) $df(0)/dI < 0$ but $f(I)$ becomes positive at some I, e.g., when $f = -a_1 I + a_2 I^2 - a_3 I^3$ where $a_1, a_2, a_3 > 0$ (Fig. 2, curve 5) and $9a_1 a_3 < 2a_2^2$, or 2) $f(I) > 0$ in the vicinity of $I = 0$ but $f(I) = o(I^2)$, e.g., $f(I) = a_1 I^3 - a_2 I^4 (a_1, a_2 > 0)$ or $f(I) = a_1 I^3/(1 + I^3/I_0^3)$, $(a_1 I_0 > 0)$, Fig. 2, curve 6. The latter nonlinearity may result from the three-photon resonant absorption of light by two-level systems with saturation.

In order to attain truly hysteretic bistable behavior (i.e., that characterized by the S-shaped steady-state curves (see, e.g., curves 2 and 3 at Fig. 1) which causes both "on" and "off" jumps between different branches of the curve), the function $f(I)$ must be positive at least in some range $0 < I < I_1$ and have a distinct peak of its first derivative df/dI in this range. The existence of hysteretic jumps is secured if $d\delta/dP = \infty$ (or $dP/d\delta = 0$) for two (or more) discrete values of P (or δ), where $dP/d\delta$ is found from (9) as

$$\frac{dP}{d\delta} = \frac{1}{2\delta}\int_0^{I_m}\left|1 - 2\frac{F(d^2F/dI^2)}{(dF/dI)^2}\right| \frac{dI}{\sqrt{\delta - F(I)}}. \quad (17)$$

A derivative $dP/d\delta$ is strongly affected by d^2F/dI^2 and therefore by df/dI; bistability may exist if $df/dI > 0$, and if at some point $I = \tilde{I}$, there is $d^2f/d\tilde{I}^2 = 0$ and $df/d\tilde{I} > df(0)/dI$. As an example of such a function, consider (Fig. 2, curve 7)

$$f = a_1 I + a_2 I^3 - a_3 I^5 \quad (18)$$

where $a_1, a_2, a_3 > 0$. S-shaped behavior of $\delta(P)$ (Fig. 1, curve 3) is possible if the condition is satisfied

$$a_1 a_3/a_2^2 < S_{cr} = 0(1) \quad (19)$$

where S_{cr} is some critical quantity; the rough estimate gives $S_{cr} \sim 0.1$–0.2. In general, the critical situation (when the curve $P(\delta)$ at some point $\delta = \delta_{cr}$ has $dP/d\delta = d^2P/d\delta^2 = 0$, see, e.g., Fig. 1, curve 4) corresponds to the conditions

$$dP/d\delta_{cr} = 0 \quad \text{and} \quad 2(d^2F/dI_{cr}^2)\, F = (dF/dI_{cr})^2 \quad (20)$$

[where I_{cr} is the minimal solution of the equation $\delta_{cr} = F(I_{cr})$], which determines both δ_{cr} and the required parameters of the function $F(I)$ [and therefore, $f(I)$]. In the case when $f(I) = 0(I^2)$ at $I = 0$, the function $\delta(P)$ forms a hysteresis if $d^2f/dI^2 > 0$, $d^3f/dI^3 > 0$, and $d^4f/dI^4 < 0$ at $I = 0$, e.g., $f = a_2 I^2 + a_3 I^3 - a_4 I^4$, $(a_2, a_3, a_4 > 0)$, see Fig. 1, curve 2. In such a case, the lower (stable) branch of $\delta(P)$ corresponds to nontrapped beam ($\delta = 0$).

To analyze the conditions (20) for the general case of an arbitrary function $f(I)$ is not an easy task, since the dependence $P(\delta)$ is implicitly determined by the integral (9), which is not evaluated in the analytic form for an arbitrary $f(I)$ [and therefore $F(I)$, (10)]. Therefore, for practical purposes, it is important to have a good analytic approximation of the function $P(\delta)$ at least in the range $0 < I < I_B$, where I_B is the point at which $d^2F(I_B)/dI^2 = 0$ (one may see from (20) that the critical point $I = I_{cr}$ is located in this range, i.e., $I_{cr} < I_B$). For such a purpose, it is more convenient to operate with I_m rather than δ [remember that $\delta = F(I_m)$]. Then, if $F(I_m)$ is positive and monotonically increasing in $(0, I_B)$, a good approximation is given by the formula

$$P(I_m) \approx 2C[\sqrt{F_m} - \sqrt{(F_m' - F_0')(F_m' - F_0' I_m)/F_m'}]/F_0' \quad (21)$$

where $F_m = F(I_m) = \delta$; $F_m' = dF(I_m)/dI$; $F_0' = dF(0)/dI$, and $C > 0$ is some constant of the order of unity which is determined by the type of function $F(I)$. For the calculations related to the conditions (20), this constant is not important, since the first of (20) is to be replaced now by the condition $dP/dI_{cr} = 0$ which makes insignificant any scaling of $P(I_m)$.

IV. Stability of Two-Dimensional Bistable Solitons

Stability of each of the possible solitons which correspond to the same total power P is an important issue. Assuming that the amplitude profile $u_s(x)$ of the particular soliton [determined by (8)] is known, we represent a solution of (2) in the form of perturbed soliton solution (6):

$$E(x, z) = [u_s(x) + \Delta u(x, z)] \exp[i\delta z/2 + i\phi(x, z)] \quad (22)$$

where Δu, $\partial\phi/\partial z$ and $\partial\phi/\partial x$ are small real perturbations. We assume factorization of the perturbation in the form

$$\Delta u = w(z)[u_s'(x)\, p(x)]; \qquad \phi = \psi(z)\, \xi(x) \quad (23)$$

where w, p, ψ, and ξ are some unknown real functions of a single variable and the prime denotes now a derivative with respect to x. Substituting (22) and (23) into (3), linearizing (3) with respect to the small perturbations, equalizing the real and imaginary parts of the obtained equation separately to zero, separating the terms that depend only on x or z, and making use of (7), one finally gets the equations for the unknown functions w, p, ψ, and ξ:

$$dw/dz = \nu\psi; \qquad d\psi/dz = \lambda w \quad (24)$$

$$(p' u_s'^2)' = \lambda (u_s^2)'\, \xi; \quad (25)$$

$$(\xi' u_s^2)' = \nu (u_s^2)' p \tag{26}$$

where ν and λ are some unknown real constants. Equations (25) and (26) can be also reduced to a single equation of the fourth order for one variable, e.g., p:

$$\{u_s^2[(p' u_s'^2)'/(u_s^2)']'\}' = \gamma (u_s^2)' p \tag{27}$$

where $\gamma = \nu\lambda$. The eigenmodes of (25)–(27) must satisfy a condition

$$p u_s' \to 0, \quad \xi'' \to 0 \quad \text{as} \quad |x| \to \infty. \tag{28}$$

If these eigenmodes as well as their respective eigenvalues ν and λ (and therefore $\gamma = \nu\lambda$) are found, the sufficient requirement for the stability of the respective soliton $E_s = u_s(x) \exp(i\delta z/2)$ is that all solutions for γ must be negative. The soliton is unstable if all of the solutions for γ are positive (the sufficient condition); the case with $\gamma = 0$ has to be further investigated although normally it corresponds to a stable soliton. The lower order eigenmodes of (25)–(27) with $\gamma = 0$ are readily found. These are as follows.

1) $\lambda = \nu = 0$, $p = \text{const} \neq 0$ (i.e., $\Delta u = C u_s'(x)$, $C << 1$), $\phi = 0$. This mode corresponds to the shift of the spatial position of the soliton peak at the x axis (with all other characteristics of the soliton being intact).

2) $\lambda = \nu = 0$, $\xi = \text{const} \neq 0$, $p = 0$, which corresponds to the shift of the soliton phase ϕ.

3) $\lambda = 0$; $\nu \neq 0$; $p = \text{const} \neq 0$; $\xi = (\nu p)x$. This corresponds to the shift of the "angle" of the soliton propagation.

4) $\lambda \neq 0$; $\nu = 0$; $\xi = \text{const} \neq 0$; and

$$p = \lambda\xi \int_0 \frac{u_s^2}{u_s'^2}\, dx = \lambda\xi \int \frac{dx}{\delta - F(u_s^2)}. \tag{29}$$

This eigenmode corresponds to a small change of the propagation constant of the soliton δ (and therefore, the total power carried by the soliton), with conservation of the single soliton character of the entire propagation. Analytical solution for the higher modes in the case of arbitrary $f(I)$ is still to be found. The detailed analysis in the case of step-nonlinearity (13) shows that the lower branch of curve 1, Fig. 1 corresponds to the unstable solitons and the upper-one to the stable ones; (the trivial solution ($\delta = 0$) is stable for any P). This suggests a general criterion for an arbitrary $F(I)$, and therefore $\delta(P)$: the stable solitons are those for which $d\delta/dP > 0$ and vice versa (see Fig. 1, curves 1–3). Although this statement seems to be intuitively almost obvious, it has still to be proven. It is also of considerable interest to study a "collision" of two solitons that belong to upper and lower branches of the curve $\delta(P)$.

V. Bistable Three-Dimensional Self-Trapping

The bistable solitons may exist also in the case of three-dimensional propagation. Spatially steady-state self-trapping of a cylindrical beam, for instance, is governed by the "nonlinear Bessel" equation which follows from (2) and (6) when $u(x, y)$ has a polar symmetry

$$d^2u/dr^2 + (1/r)(du/dr) + u]f(u^2) - \delta] = 0 \tag{30}$$

where $r = \sqrt{x^2 + y^2}$ is a radial coordinate in the plane normal to z axis. With the step-nonlinearity (13) the solution of (30) is

$$u(r) = \begin{cases} AJ_0(kr\sqrt{\Delta - \delta}), & \text{when} \quad u > u_0 \equiv \sqrt{I_0}, \quad r < r_0 \\ BK_0(kr\sqrt{\delta}), & \text{when} \quad u < u_0, \quad r > r_0 \end{cases} \tag{31}$$

where J_ν and K_ν are ν-order ordinary and modified Bessel functions respectively; A, B, and r_0 are some constants. By requiring continuity of $u(r)$ and du/dr at $u = u_0$, one gets an equation determining the radius of the beam r_0 at the level $u = u_0$:

$$\sqrt{\frac{\Delta}{\delta} - 1}\, \frac{J_1(kr_0\sqrt{\Delta - r})}{J_0(kr_0\sqrt{\Delta - r})} = \frac{K_1(kr_0\sqrt{\delta})}{K_0(kr_0\sqrt{\delta})}. \tag{32}$$

With r_0 known, A and B are determined as $A = u_0/J_0(kr_0\sqrt{\Delta - \delta})$; $B = u_0/K_0(kr_0\sqrt{\delta})$. For cylindrical beams, a Kerr-nonlinearity, $f \propto I$, plays the same role as $f \propto I^2$ in the two dimensional case: for such a nonlinearity, the total power of the beam does not depend of its size or its peak intensity [3]. Therefore, in order to attain a nonhysteretic bistable soliton propagation of the kind depicted by curve 1, Fig. 1, the lower required degree of nonlinearity at $I \to 0$ is $f \propto I^2$ [with f attaining some maximum or saturation when I increases, e.g., $f = \alpha I^2/(1 + I^2/I_0^2)$] which resembles curve 6, Fig. 2. Such a nonlinearity can be originated, e.g., by the two-photon resonant absorption [16]. An S-shape hysteretic characteristic curve $\delta(P)$ can be provided now by the nonlinearity $f(I) = a_1 I + a_2 I^2 - a_3 I^3$ ($a_1, a_2, a_3 > 0$) which resembles curve 7, Fig. 2, with the critical condition in the same form as (10) [but with different $S_{cr} = 0(1)$].

VI. Conclusion

In conclusion, we demonstrated an existence of multistate solitons of generalized nonlinear Schrödinger equation. In order for those solutions to exist, the nonlinearity must satisfy some special conditions, e.g., its dependence on the light intensity must have a range where it increases sufficiently sharply. In nonlinear optics, these solitons may manifest themselves either as single multistate pulses (e.g., in nonlinear fibers) or self-trapped multistate channels (both in two- and three-dimensional cases). Multistate solitons present the ultimate case of multistable wave propagation and may find an application to the dynamic (temporal) optical bistability and bistable resonator-free self-trapping of light.

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A. E. Kaplan, for a photograph and biography, see this issue, p. 1536.