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"Robust" bistable solitons of the highly nonlinear Schrödinger equation

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We found that for some highly nonlinear Schrödinger equations (as contrasted to the cubic equation) the criteria of stability of solitary waves against small and large perturbations do not coincide, which results in the existence of "weak" and "robust" solitons, respectively. We have shown that bistable solitons, predicted earlier by Kaplan [Phys. Rev. Lett. **55**, 1291 (1985)], are robust for some particular nonlinearities and, therefore, physically feasible. We have also suggested a general criterion for robustness of solitons.

Solitons, by definition, are *stable* solitary solutions of nonlinear wave equations. These equations and their solitons result from many problems in the theory of elementary particles, nonlinear optics and electrodynamics, plasma physics, hydrodynamics, biology, etc. It is well known, for example, that the nondegenerate solitary-wave solutions of the cubic nonlinear Schrödinger equation (which has many applications in nonlinear optics) are stable against both small and large perturbations; in particular, two such singular solitary waves survive their collision, with their individual energies and momenta conserved after the collision,[1] i.e., they are solitons. Since the cubic nonlinear equation is the most famous of the nonlinear Schrödinger-like equations studied so far, the distinction between these two types of stability has never been, to the best of our knowledge, clearly drawn in the literature. Very often for such (and other)[2] nonlinear equations, the conditions of stability from small-perturbation analysis are automatically regarded as universal criteria for soliton existence. However, this issue becomes increasingly important, especially in application to highly nonlinear Schrödinger equations with their functions of nonlinearity drastically different from the simplest known cubic nonlinearity. It has been recently demonstrated by Kaplan[3] that for a certain class of nonlinearities, bistable (and, in general, multistable) solitary waves can exist which carry the same energy but have distinctly different profiles and speeds of propagation. The issue of the stability of these new solutions is of prime importance for their physical feasibility.

In this Rapid Communication we show that the stability against small perturbations alone (as well as instability against large perturbations alone) *does not* provide a comprehensive description of stability of solitary-wave solutions of highly nonlinear Schrödinger equations. In order to explore this issue, we introduce here the notion of "robust" solitons as distinct from "weak" solitons in the sense that the latter are stable against (sufficiently) small perturbations, whereas the former are stable against any possible perturbation, including large perturbations; in particular, perturbations in the form of collisions with other solitary waves. (Solitons of a cubic Schrödinger equation are robust in this sense.) By studying a wide variety of nonlinear models, many of which exhibit bistability for certain ranges of the parameters, we have found nonlinear models which exhibit robust (in particular, *bistable* robust) solitons as well as models which exhibit weak solitons. Some of them also have solitary solutions which are unstable against any perturbations. We shall further suggest a general criterion for "robustness," which is valid for arbitrary nonlinear models. The fact that "robust" *bistable* solitons of the highly nonlinear Schrödinger equation are possible may prove to be of significant importance for such nonlinear optical applications as fiber-optics communication with undistorted pulses,[4] compresssion of optical pulses,[4,5] and optical switching and bistability,[6] and may prove to stimulate the search for the materials and mechanisms with appropriate nonlinearities. Examples of such higher-order nonlinear mechanisms are light-induced phase transitions and multiphoton resonances (see, e.g., Ref. 3).

The generalized nonlinear Schrödinger equation[3] for the complex electric field amplitude E for one-dimensional pulse propagation is

$$2i\partial E/\partial z+\partial^2 E/\partial x^2+Ef(|E|^2)=0 \;, \tag{1}$$

where $f(|E|^2)$ is an arbitrary function of the intensity $I\equiv|E|^2$. Here $z\propto z_1$, z_1 being the distance coordinate in the direction of propagation, while x is proportional to $t-z_1/v_g$, t being the time variable and v_g the group velocity. The same equation governs two-dimensional self-trapping.[3] The first three invariants (i.e., quantities independent of z) of Eq. (1) are found to be

$$\begin{aligned} J_1&=\int_{-\infty}^{\infty} I\,dx \;, \\ J_2&=i\int_{-\infty}^{\infty}(EE_x^*-\text{c.c.})dx \;, \\ J_3&=\int_{-\infty}^{\infty}\left[|E_x|^2-\int_0^I f(s)ds\right]dx \;, \end{aligned} \tag{2}$$

corresponding to conservation of the total power, total "transverse" momentum, and "transverse" energy of the field, respectively; they are consistent with the first three

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conservation laws[1] for the cubic Schrödinger equation, when $f(I) \propto I$. We used J_1 and J_3 to check the numerical accuracy of our computer simulations. For the purpose of studying collisions, solitary-wave solutions of Eq. (1) of the form

$$E(x,z) = U(x - wz)\exp(i\delta z/2 + iwx)$$

are sought, where $U = |E|$ is a real function satisfying the condition $U \to 0$ as $|x| \to \infty$, δ is a real constant, and w is playing the role of a (real) velocity parameter. The resulting equation for U is of the form

$$d^2U/dx^2 + U[f(U^2) - \bar{\delta}] = 0 \ , \qquad (3)$$

with $\bar{\delta} \equiv \delta + w^2$. All of the analytic results obtained earlier by Kaplan[3] now follow with δ replaced by $\bar{\delta}$. We now concentrate on the stability issue, particularly for those $f(I)$ which yield bistable solitons. We designate the total power J_1, for the particular case of a solitary solution as $P = \int_{-\infty}^{\infty} U^2 dx$, where $P(\bar{\delta})$ can be determined[3] directly without explicitly solving Eq. (3) for $U(x)$. Bistability occurs when the P vs $\bar{\delta}$ curve is "N shaped" or, equivalently, $\bar{\delta}(P)$ is "S shaped." As illustrative examples of models displaying bistability with both robust and weak solitons, we consider here (1) the polynomial model $f = a_1 I + a_2 I^3 - a_3 I^5$ with $a_1, a_2, a_3 > 0$ and (2) the "linear + smooth-step" model, $f = aI$ for $I \le I_0$, $\Delta[1 - (1-\mu) I_0^2/I^2]$ for $I \ge I_0$ with $a, \Delta > 0$, $\mu = aI_0/\Delta$, and $0 < \mu < 1$. The first model corresponds to Eq. (13) of Ref. 3(a), while the second is a generalization of Eq. (10) of the same reference. These two models will illustrate the central points and conclusions reached from studying[7] a large number of nonlinear functions $f(I)$.

(1) The polynomial model. To build a polynomial model with bistability present, one should note that for $f = aI^n$, with a and $n > 0$ one can *analytically* show that the power $P \sim (\bar{\delta})^{1/n - 1/2}$. For $n = 1$ (cubic Schrödinger case), $dP/d\bar{\delta} > 0$, for $n = 3$ it is negative, etc. For $a < 0$ the opposite sign applies. Thus, for the polynomial model $f = a_1 I + a_2 I^3 - a_3 I^5$, an N-shaped $P(\bar{\delta})$ curve will result by suitably adusting a_1, a_2, and a_3 with each of the three branches of the N associated with one of the terms in $f(I)$. For this model, P and $U(x)$ *cannot* be obtained analytically. To obtain the solitary profiles, Eq. (3) was integrated numerically for different values of $\bar{\delta}$ using a standard fourth-order Runge-Kutta scheme. The area under the U^2 curve then gave us P as a function of $\bar{\delta}$. For a given value of the ratio $R = a_1 a_3 / a_2^2$, a universal power curve results if $(a_1 \sqrt{a_3/a_2})^{1/2} P \equiv \rho$ is plotted against $[\sqrt{(a_3/a_2)}/a_1]\bar{\delta}/\bar{\delta}_\infty \equiv \beta$, where $\bar{\delta}_\infty$ is the value of $\bar{\delta}$ at which $dP/d\bar{\delta} \to \infty$. Bistability was found to occur for $R \le 0.08$, Fig. 1 showing the $\rho(\beta)$ curve for $R = 0.04$. In this case $\bar{\delta}_\infty = 2.62$. Lowering the ratio R increases the size of the "dip" in the $\rho(\beta)$ curve but does not affect the conclusions reached about stability. Kaplan[3] had suggested a general stability criterion for arbitrary $f(I)$, namely, that stable solutions should correspond to those portions of the $\rho(\beta)$ curve for which $d\rho/d\beta > 0$ (or $dP/d\bar{\delta} > 0$) and vice versa. He further suggested that in the context of bistability, the stability should be tested by allowing solitary pulses with the same value of ρ but different values of β to collide and see whether the pulses emerged unchanged or not.

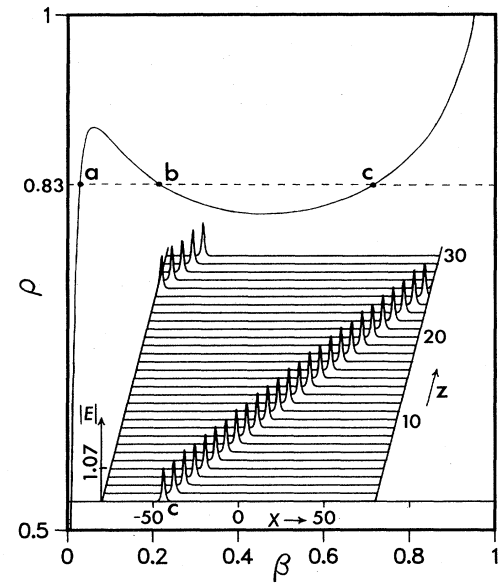

FIG. 1. Normalized power ρ vs the parameter β for the polynomial model for $R = 0.04$ and $\bar{\delta}_c = 1.5$. The points labeled *a*, *b*, and *c* correspond to three different possible solitary-wave solutions with the same $\rho \simeq 0.83$. Inset shows stability of the solitary-wave solution corresponding to *c* against a (sufficiently) small perturbation. Mesh size: $\Delta x = 0.08$ for all plots, $\Delta z = 5 \times 10^{-4}$. The pulse reappears on the opposite edge due to the use of periodic boundary condition.

In a preliminary numerical study, Enns and Rangnekar[8] had found that this stability criterion was satisfied for the smooth-step model $f = \Delta(1 - I_0^2/I^2), I \ge I_0$, and zero for $I \le I_0$ proposed by Kaplan,[3] this model having a stable upper branch for $\beta \equiv \bar{\delta}/\Delta \ge \beta_{cr} = 0.15$ and an unstable lower branch for $\beta < \beta_{cr}$. From the bistability viewpoint, a more important test[9] is to study the collision between two solitary pulses belonging to the upper and lower branches of a model (e.g., the polynomial model) for which *both* branches have $d\rho/d\beta > 0$. In Fig. 1, the three points labeled *a*, *b*, and *c* correspond to different β values (0.027, 0.218, and 0.724, respectively) with the same $\rho(\simeq 0.83)$. According to the above criterion, solitary-wave profiles corresponding to *a* and *c* should remain stable under collision, but the one corresponding to *b* should be unstable.

To study the collision process, Eq. (1) was simulated numerically, using the same explicit scheme with periodic boundary conditions as in Refs. 8 and 10. Although different speeds could be (and were) assigned to colliding solitary waves, for convenience $|w| = 5$ for the pulses in all plots presented here. In the lab frame $w > 0$ (< 0) corresponds to a velocity less (greater) than v_g. The numerical accuracy was determined by monitoring the invariants J_1 and J_3, Eq. (2). In all our computer runs, $\max|\Delta J_1/J_1|$ was better than 1.5% and $\max|\Delta J_3/J_3|$ better than 5%.

Figure 2 shows the collision between solitary pulses corresponding to the points *a* and *c* in Fig. 1. The smaller pulse (a) remains unchanged after the collision; however,

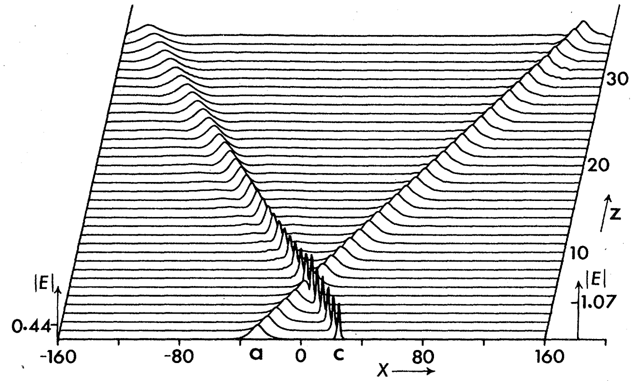

FIG. 2. Collision between solitary waves corresponding to points a (pulse initially on left) and c (on right) of Fig. 1 with $R=0.04$, $\bar{\delta}_a=0.056$, $\bar{\delta}_c=1.5$, $\Delta z=10^{-3}$.

the larger pulse (c) clearly disperses, i.e., is not stable against collision with the smaller pulse. When pulses corresponding to b and c collide (not shown), both pulses were observed to completely flatten out after the collision, the behavior being expected for the middle one, but again, not for the larger pulse. All combinations of possible collisions and ratios R were considered, the general conclusion being that solitary pulses belonging to the upper (positive-slope) branch are *not* stable against large perturbations (in the form of collisions) even though $d\rho/d\beta>0$. A similar unstable behavior had been observed for sufficiently large pulses by Cowan, Enns, Rangnekar, and Sanghera[10] (see their Fig. 4) for the nonlinear model $f=a_1I+a_2I^2$ ($a_1,a_2>0$) for which $dP/d\bar{\delta}$ is positive everywhere (no bistability present). What about stability against (sufficiently) small perturbations? The inset of Fig. 1 shows that for a sufficiently small perturbation (in this run just numerical noise, but also valid for collision with sufficiently small pulses) the solitary pulse corresponding to c is *stable.* Similarly, the solitary-wave profile of Fig. 4 of Ref. 10 was also found to be stable against a small perturbation (even against non-negligible perturbations). Therefore, all of them are weak solitons. Indeed, for all models studied, the following was found to be universally true: *$dP/d\bar{\delta}>0$ guarantees stability against (sufficiently) small perturbations. It is a necessary but not sufficient condition for robustness of a soliton. On the other hand, $dP/d\bar{\delta}<0$ guarantees unconditional instability*[3,8] *(as will be illustrated for the next model).*

Since the question remains unanswered about whether one can ever have stability against large perturbations for two solitary waves with the same power and *both* with $dP/d\bar{\delta}>0$, we returned to the smooth-step model introduced in Ref. 3 and studied for stability in Ref. 8. We reasoned that since in this model the upper branch was already stable against collisions in our numerical runs, perhaps it would be possible to splice a lower stable branch onto the smooth-step model. This led us to the linear + smooth-step model, the point being that $f\propto I$ (for arbitrary I) yields the cubic nonlinear Schrödinger equation with known soliton solutions. It should be noted, however, that modification[7] of this model by including a quadratic term in I for $I\leq I_0$ did not alter the conclusions.

(2) The linear + smooth-step model. For this model, Eq. (3) is exactly integrable,[7] yielding solitary-wave profiles in terms of hyperbolic and trigonometric functions. The power formula, $P=P(\bar{\delta})$ [or $\rho=\rho(\beta)$, with $\rho\equiv P\sqrt{\Delta}/I_0$ and $\beta\equiv\bar{\delta}/\Delta$] is also readily found. For $\mu=0$, these formulae reduce to those of the smooth-step model in Refs. 3 and 8. The curves in Fig. 3 show how ρ varies as a function of μ. Bistability occurs for $\mu\lesssim0.42$. Again we can select three points for the same ρ, e.g., a, b, and c on the $\mu=0.1$ curve with $\beta=0.031$, 0.065, and 0.4, respectively, all corresponding to $\rho\simeq6.97$. Figure 4(a) shows a collision between pulses corresponding to c (the large pulse) and a (the small pulse). *Both* pulses, after collision, appear to be identical to the input profiles, i.e., they are quite stable. Further, like solitons of the cubic Schrödinger equation,[1] the pulses corresponding to a and c are found to be able to survive collisions with themselves; they are typical robust solitons (which is also confirmed by simulations of collisions with large nonsoliton pulses).[7] An example of this robust behavior is given in Fig. 4(b), where two large pulses corresponding to c remain unchanged after the collision. The pulse corresponding to b, on the other hand, was found to be unstable against collisions with pulses corresponding to a and c. Indeed, any point on the $\rho(\beta)$ curves with $d\rho/d\beta<0$ was found to be unstable against infinitesimal perturbations as can be seen in the inset of Fig. 3, where the solitary wave corresponding to the point b' ($\beta=0.032$) on the $\mu=0.05$ curve begins to radiate after colliding with a second pulse too small ($\beta=10^{-12}$) to be seen in the plot. For all combinations of collisions and all μ values (including $\mu>\mu_{cr}$), it was found

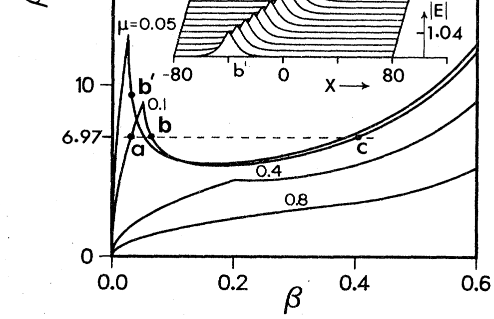

FIG. 3. Normalized power ρ vs β for the linear + smooth-step model for different μ values. The points a, b, and c on the $\mu=0.1$ curve *all* correspond to $\rho\simeq6.97$. Inset shows instability against collision of a solitary wave corresponding to point b' on the $\mu=0.05$ curve with a solitary pulse too small ($\beta=10^{-12}$) to be seen in the plot. Parameters: $\Delta=I_0=1$, $\Delta z=3\times10^{-3}$.

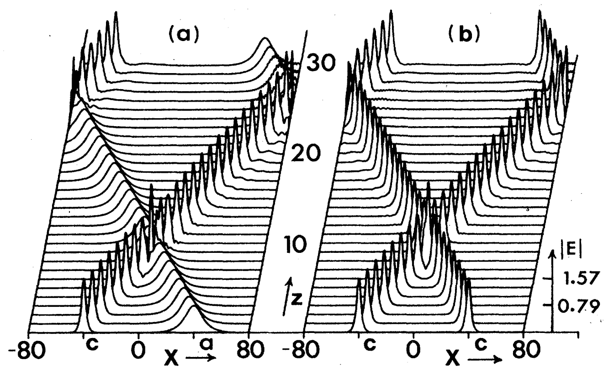

FIG. 4. Typical collision results for the linear + smooth-step model. (a) For solitary waves corresponding to points a and c on the $\mu=0.1$ curve of Fig. 3. Pulse c is initially on the left, a on the right. (b) For two solitary waves corresponding to point c on the $\mu=0.1$ curve of Fig. 3.

that those pulses corresponding to $dp/d\beta > 0$, were stable against collisions, i.e., it is possible to create a nonlinear model for which robust bistable (for $\mu < \mu_{cr}$) solitons are possible.

Results such as these raise the question of what general class of bistable models can robust solitons occur for. Consistent with the above models and many others that we have studied,[7] we would suggest the following criterion for robustness. Solitary waves are robust solitons (i.e., stable against both small and large perturbations) if (a) $dP/d\bar{\delta} > 0$, stability against (sufficiently) small perturbations (this necessary condition has already been discussed at length), (b) $f(I)/I^2 = o(1)$ as $I \to \infty$, stability against collapse ("self-focusing"), and (c) $f(I)$ is a non-negative and nondecreasing function for $I > 0$, stability against dispersion.

Condition (b) prevents the occurrence of "self-focusing singularities" which occur[11] for $f \sim aI^n$ $(a > 0)$ for large I for $n \geq 2$. It excludes, e.g., the model $f = a_1 I + a_2 I^2$ $(a_1, a_2 > 0)$ for which explosive behavior occurred.[10,11] Condition (c) is suggested as a sufficient condition and may be somewhat stronger than is actually needed; we introduced it to rule out dispersive models such as the polynomial model discussed above and the model with $a_2 < 0$ examined by Cowan *et al.*[10] For the latter, "quasisoliton" behavior was observed in the collision process but, in general, accompanied by an appreciable radiative peak or background. Radiation is associated with dispersive effects. If $f(I)$ decreases sufficiently at large I, then the derivative terms of Eq. (1) will predominate and it is well known that these terms tend to disperse or spread pulses.

In conclusion, for highly nonlinear Schrödinger equations we have (i) demonstrated the necessity to distinguish between stability against small and large perturbations, (ii) introduced the concept of robustness for solitons and suggested a criterion for robustness, and (iii) most importantly shown that *bistable* robust solitons are possible.

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