Optical high-order subharmonic excitation of free cyclotron electrons

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Homogeneous laser radiation in the visible or infrared range can excite high-order subharmonics at the cyclotron frequency of free electrons in the millimeter or microwave range. This may provide coherent links between lasers and rf or microwave frequency standards. In order to divide the frequency of a CO_2 laser ($\lambda \approx 10\ \mu m$) by a factor of 100 down to $\lambda \approx 1$ mm in one step, a cw laser power as low as 10^{-6} W is sufficient.

The interaction of electromagnetic (EM) waves with free electrons can result in strong nonlinear-optical effects[1–3] based on relativistic phenomena. It was recently predicted[1] that the cyclotron motion of a single electron must demonstrate a strongly hysteretic resonance based on a small relativistic change of mass; consistent with this prediction, this effect was subsequently observed in the experiment[2] at the main frequency (i.e., when $\omega \approx \Omega_c$, where ω is the frequency of the driving field and Ω_c is the cyclotron frequency). The free-electron system, which is probably the most fundamental nonlinear system,[4] must also be capable of demonstrating many other features of nonlinear oscillators. It is feasible,[3] for example, to obtain strong hysteretic multiphoton excitation of a cyclotron electron driven by biharmonic laser radiation with optical frequencies ω_1 and ω_2 such that $\omega_1 - \omega_2 \approx \Omega_c$ (three-photon resonance) or $\omega_1 - \omega_2 \approx 2\Omega_c$ (four-photon resonance).

In this Letter we show that yet another multiphoton effect with a driven cyclotron electron is feasible, which consists in the generation of high-order subharmonics $\Omega \approx \Omega_c$ of a driving laser (optical) field with frequency ω such that $\omega = n\Omega$, n (≥ 2) being an integer.

The subharmonic excitation of cyclotron motion by rf or microwave (mw) driving sources is well known; in fact, the synchrotron[5] and synchrocyclotron[6] principles of particle acceleration are based on driving a particle beam at the frequency equal to the multiplied cycling frequency of accelerated particles. An analogous principle was recently proposed[7] to obtain mw subharmonic radiation of electrons by using a laser as a driving source. In both mw accelerators[5,6] and laser[7] schemes the important condition is that the driving field acts upon particles at a distance much shorter than the orbit circumference. In this Letter we show that, because of the Doppler and Lorentz nonlinear mechanisms,[3] the high-order subharmonic excitation is feasible even when the driving field is a plane standing (or traveling) wave acting upon a particle along its entire cyclotron orbit. In order to obtain the 100th-*order subharmonic of a* CO_2 *laser* ($\lambda \approx 10\ \mu m$), i.e., to divide its frequency by a factor of 100 down to $\lambda = 1$ mm in one step, a cw power as low as $\approx 10^{-6}$ W is sufficient.

It is well known in the theory of synchrotron radiation[8] that a relativistic cyclotron electron emits radiation not only at the cyclotron's frequency but also at its higher harmonics.[8,9] In comparing the subharmonic excitation with the generation of higher harmonics, one has to keep in mind that the former process is much more complicated (for instance, the subharmonics are characterized by a threshold excitation, by the existence of multiple equidistant states of phase, etc.; see below).

High-order subharmonics have been observed in many resonant nonlinear systems. For example, in a simple nonlinear circuit using a biased diode as a nonlinear capacitor, a one-step frequency division by a factor up to 500–1000 in the ultrahigh rf range has been observed and studied.[10] The subharmonics in Ref. 10 were attributable to the self-synchronization of so-called parametric oscillations induced by a driving force. The same principle was later proposed[11] to obtain low-order subharmonics in an optical range by using an optical parametric oscillator.

The quest for optical one-step multiple coherent transformation of frequency stems from the need to cover a gap between optical and mw time and frequency standards. The conventional techniques are based on the frequency multipliers,[12] complex frequency synthesis chains,[13] frequency division involving locking both a laser and a rf source to a cavity,[14] etc. The proposed high-order cyclotron subharmonics have the potential to provide a promising alternative method for obtaining a direct coherent link between lasers and mw frequency standards.

Consider a free electron in a homogeneous magnetic field $\mathbf{H}_0 = H_0\hat{e}_z$ (such that the unperturbed cyclotron frequency is $\Omega_0 = eH_0/m_0c$). The electron is driven by two counterpropagating plane waves,

$$\mathbf{E}_j = \hat{e}_y E \cos[\omega t + (-1)^j(kx - \psi)], \quad j = 1, 2, \quad k = \omega/c, \tag{1}$$

with the same frequency ω and amplitude E. Both of the waves propagate along the axis x normal to $\mathbf{H}_0$, are polarized along the axis y, also normal to $\mathbf{H}_0$ (see insert in Fig. 1), and form a standing-wave pattern.[15] We introduce a dimensionless radius vector of an electron,

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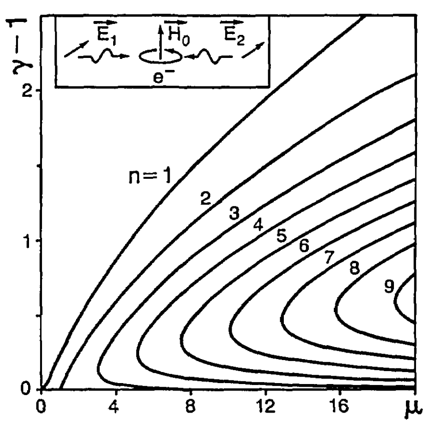

Fig. 1. Extremum kinetic energy $\gamma - 1$ of the cyclotron motion versus driving parameter μ for various orders of subharmonics n. Insert: incident light configuration with respect to the dc magnetic field $\mathbf{H}_0$.

$\zeta = \mathbf{r}\Omega_0/c$; its momentum, $\rho = \mathbf{p}/m_0c$; its energy, $\gamma = \sqrt{1 + \rho^2}$ (with $\rho = \gamma\Omega_0^{-1}\mathrm{d}\zeta/\mathrm{d}t$); and fields $f_j = eE_j/m_0c\Omega_0$. The Lorentz equation is then written in the form

$$\Omega_0^{-1}\mathrm{d}\rho/\mathrm{d}t + \Gamma\gamma\rho = \hat{e}_y(f_1 + f_2) + \gamma^{-1}[\rho \times \hat{e}_z] \times [1 + (f_1 - f_2)] + \mathbf{g}(\zeta), \quad (2)$$

where $\Gamma\gamma\rho$ is a damping term[3] that is due to the synchrotron radiation, with $\Gamma = 2e^2\Omega_0/3m_0c^3$ and $f_j = f_j\{\omega t + (-1)^j[\hat{e}_x \cdot \zeta(t) - \psi]\}$, the term with $(f_1 - f_2)$ corresponds to the Lorentz force of the EM field, and $\mathbf{g}$ represents a (weak) trapping electrostatic potential. When the subharmonic of the nth order is excited, the cyclotron motion of the electron can be described as

$$\zeta(t) = \zeta_c[\hat{e}_x \sin(\Omega t + \phi) + \hat{e}_y \cos(\Omega t + \phi)] + \zeta^{(0)}, \quad (3)$$

where $\Omega = \omega/n$. The radius ζ_c of the cyclotron motion, its phase ϕ, and radius vector $\zeta^{(0)}$ of the center of orbit are assumed to be slowly varying functions of time. In our further calculations we assume that the center of orbit is located in the center of a trap and coincides with the origin, i.e., always $\zeta^{(0)} = 0$. The center of orbit must also coincide with a maximum of the total field $\mathbf{E}_1 + \mathbf{E}_2$ [which corresponds to $\psi = 0$ in Eq. (1)] when n is odd and with a node of the total field (i.e., $\psi = \pi/2$) when n is even.

The nonlinear processes attributable to the excitation of cyclotron motion and sustained subharmonic oscillation when $\omega > \Omega_c$ or even when $\omega \gg \Omega_c$ are (1) the fast modulation of phase $\hat{e}_x \cdot \zeta(t)$ of fields f_j in Eq. (2) seen by a rotating electron (which is essentially the Doppler effect) and (2) the Lorentz force of the EM field [the term related to $f_1 - f_2$ in Eq. (2)]. Inserting Eq. (3) into Eq. (2), retaining only the terms with the cyclotron frequency Ω, and keeping in mind that f, $\Gamma \lll 1$ (e.g., at $\lambda_0 \approx 1$ mm, $\Gamma \approx 10^{-11}$), we obtain the equations governing the dynamics of the cyclotron orbit radius ζ_c and the phase of the rotating electron ϕ:

$$\frac{1}{\Omega_0}\frac{\mathrm{d}\zeta_c}{\mathrm{d}t} + \Gamma\gamma\zeta_c = -f\sin(n\phi - \psi) \times [J_{n-1}(n\beta) - J_{n+1}(n\beta)], \quad (4)$$

$$\frac{1}{\Omega_0}\frac{\mathrm{d}\phi}{\mathrm{d}t} + \left(\frac{\omega}{n\Omega_0} - \frac{1}{\gamma}\right) = -\frac{2}{\beta^2\gamma^3} f\cos(n\phi - \psi)J_n(n\beta), \quad (5)$$

where $\beta = \sqrt{1 - \gamma^{-2}} = \zeta_c/\sqrt{1 + \zeta_c^2}$ and J_ν is an ordinary Bessel function of the νth order. In the equilibrium state ($\mathrm{d}/\mathrm{d}t = 0$), the energy of the excitation is (see Fig. 2)

$$\gamma = n\Omega_0/\omega + \Delta\gamma \qquad (\text{and } \zeta_c = \rho = \gamma\sqrt{1 - \gamma^{-2}}), \quad (6)$$

with $\Delta\gamma = 0(\Gamma) \lll 1$; for $n \geq 2$

$$\frac{\Delta\gamma}{\Gamma} = \pm\frac{2J_n(n\beta)}{\beta^2\gamma}\left\{\mu^2 - \frac{\beta^2\gamma^4}{[J_{n-1}(n\beta) - J_{n+1}(n\beta)]^2}\right\}^{1/2}, \quad (7)$$

where we introduced a driving parameter $\mu = f/\Gamma = 3Ec^2/2e\Omega_0^2$. According to Eq. (6), the frequency of subharmonic ω/n follows closely the effective cyclotron frequency $\Omega_c = \Omega_0/\gamma$ determined by the relativistic mass effect $m/m_0 = \gamma$. The stability analysis of Eqs. (4) and (5) shows that the upper sign in Eq. (7) corresponds to stable states and the lower sign to unstable states. It is obvious from Eqs. (4) and (5) that for each stable magnitude of the energy γ (and therefore for each orbit radius $\zeta_c = \gamma\sqrt{1 - \gamma^{-2}}$), the cyclotron motion can have n equally possible *different equidistant* states of the phase ϕ, $\phi = \phi_0(n, f, \omega) + 2\pi s/n$ [s being an integer, $0 \leq s < n$, and ϕ_0 determined, e.g., by Eq. (4) with $\mathrm{d}/\mathrm{d}t = 0$]. This property is common for *any subharmonic oscillations of the nth order* regardless of their origin.[10]

For any order $n > 2$, there are both *upper* and *lower* limits of the energy $\gamma = (1 - \beta^2)^{-1/2}$ of the excited subharmonic for any fixed driving amplitude f (or μ) that are determined by the relationship (see Fig. 1)

$$\mu = \beta/(1 - \beta^2)[J_{n-1}(n\beta) - J_{n+1}(n\beta)], \quad (8)$$

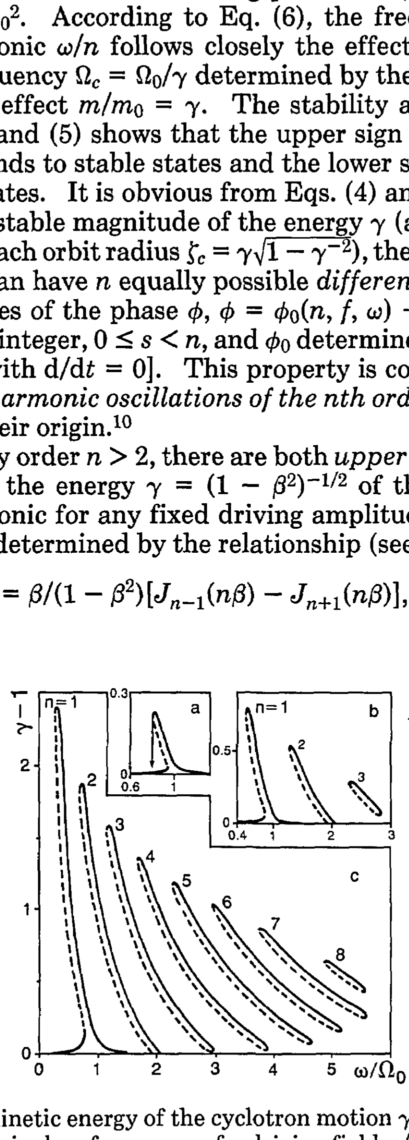

Fig. 2. Kinetic energy of the cyclotron motion $\gamma - 1$ versus the dimensionless frequency of a driving field ω/Ω_0 for various orders of subharmonics n ($n = 1$ is the main resonance, $n = 2$ is the second subharmonic, etc.) and for the fixed amplitude parameter $\mu = f/\Gamma = E/E_0$ of a driving field. a, $\mu = 0.9$; b, $\mu = 3.5$; c, $\mu = 16$. The thick solid branches correspond to stable states; the thick broken branches to unstable states. All the branches are stretched out along lines determined by the formula $\gamma = n\Omega_0/\omega$ for each order n.

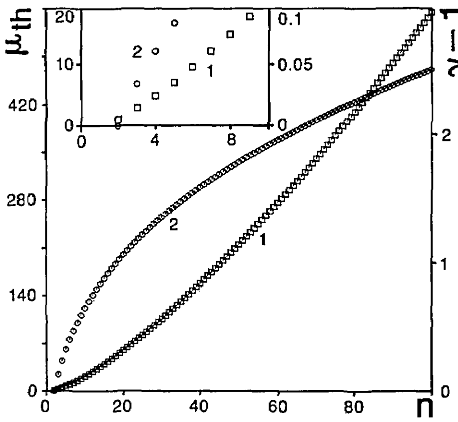

Fig. 3. Threshold driving parameter $\mu_{th} = E_{th}/E_0$ (curve 1) and kinetic energy $\gamma - 1$ corresponding to μ_{th} (curve 2) versus order of subharmonic n. Insert: magnified plot in the vicinity of the origin.

which follows from Eq. (7). This gives rise to the formation of isolated branches of excitation (so-called isolas[16]) for each individual subharmonic with $n > 2$. The initial start jump required for the electron to reach the desirable subharmonic can be provided by triggering the system either with the microwave oscillator having its cyclotron frequency (or some of its higher harmonics) near the cyclotron frequency with injection of preaccelerated electrons or with optical biharmonic pumping[3] by a laser having $\omega_1 - \omega_2 = \Omega$. It also follows from Eq. (8) that for any $n \geq 2$, there is a *minimum (threshold) driving amplitude* μ_{th}, *which is* an increasing function of n (see Fig. 3, curve 1) with μ_{th} $(n = 2) = 1$. The threshold amplitude $\mu_{th}(n)$ corresponds to some certain energy of excitation (curve 2 in Fig. 3) that increases (although slowly) as n increases. However, as the driving amplitude μ exceeds the threshold level μ_{th} for any particular subharmonic, the minimal limit of the kinetic energy for that subharmonic rapidly decreases, such that the system can be operated at reasonably low energies and the frequency of the subharmonic comes close to the unperturbed cyclotron frequency.

The intensity of driving laser radiation required to excite even high-order subharmonics is strikingly low. Indeed, the threshold amplitude to excite the second-order subharmonic ($\mu_{th} = 1$) is

$$E_0 = 2e\Omega_0^2/3c^2 = (2/3)(m_0c^2/e)k_0^2r_e, \tag{9}$$

where $k_0 = \Omega_0/c$ and $r_e = e^2/m_0c^2 = 2.8 \times 10^{-13}$ cm is a classic radius of an electron. For the cyclotron wavelength $\lambda_c \approx 1$ mm, E_0 corresponds to as low intensity as $\approx 2 \times 10^{-10}$ W/cm^2. Consider now an example when the laser wavelength is $\lambda = 10$ μm (CO_2 laser), whereas $\lambda_c = 1$ mm, and therefore $n = 100$, i.e., the laser frequency is divided by a factor of 100 in one step. According to Fig. 3, curve 1, the threshold parameter for $n = 100$ is $\mu_{th} \approx 560$. We choose the driving force μ four times above this threshold to make sure that the *minimal energy of the 100th subharmonic is low and* therefore that the oscillation frequency does not differ much from the unperturbed cyclotron frequency. The resulting intensity will still be very low, $\approx 10^{-2}$ W/cm^2. With the area of the beam $\lambda \times \lambda_c \approx 10^{-4}$ cm^2 this translates into the total driving power as low as 10^{-6} W.

In conclusion, we have demonstrated the feasibility of obtaining very high-order (up to a few orders of magnitude) subharmonics of the driving laser frequency by using free electrons in a magnetic field with a cyclotron frequency in the microwave range. This subharmonic excitation is attributable to the combination of the Doppler effect, the Lorentz force, and the relativistic change of mass. It can be obtained with almost homogeneous radiation; no concentration of a laser beam at a small portion of the electron's orbit is required, and virtually any cw infrared laser can be used as a driving source.

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15. It is worth noting that a standing-wave configuration is used in this Letter for the sake of symmetry and calculation simplicity. In fact, subharmonic oscillations can be obtained by using a plane traveling wave instead of a standing wave.
16. See, e.g., A. E. Kaplan and C. T. Law, IEEE J. Quantum Electron. **QE-21**, 1529 (1985).