JOURNAL DE PHYSIQUE
Colloque C2, Supplément au n°6, Tome 49, juin 1988

BISTABLE OPTICAL SOLITONS

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Abstract

The generalized nonlinear Schrödinger equation with certain nonlinearities allows for the existence of multi-stable single solitons (i.e., singular solitons with the same carried power but different profiles and propagation parameters). Some of these new solitons are absolutely unstable, whereas the rest fall into two classes of either "weekly" (i.e. stable against small perturbation) or absolutely stable solitons (the so called "robust" solitons that are stable against arbitrary perturbation, in particularly in the form of collision with another large soliton). The criteria for both weak stability and robustness are suggested and tested in computer simulations for various models of nonlinearity. In nonlinear optics, these solitons may exist either in the form of short bistable pulses, or bistable self-trapping (both two- and three-dimensional). The recent research shows that the bistable solitons can be switched from one stable branch to another and that the originating nonlinear equation passes the Painleve test for complete integratibility.

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It was recently demonstrated by us [1] that for a certain class of nonlinearities, the soliton solution of the (generalized) nonlinear Schrödinger equation becomes multi-stable. This implies that more than one amplitude profile and speed of propagation of a singular soliton may exist for the same amount of the total power carried by the soliton. The existence of multi-stable solitons is related to the type of dependence of nonlinear susceptibility on the intensity of light. For example, the multistable solitons waves cannot be observed in a Kerr-like nonlinear medium; they may exist only if the nonlinear component of the susceptibility as function of intensity is either changing its sign or its derivative has a sufficiently sharp peak (e.g. is a step-like function). Most recent research [2–5] by Enns, Rangnekar, and Kaplan confirmed that they are indeed solitons having new and very interesting properties.

Consider the generalized nonlinear Schrödinger equation for the complex amplitude of field E in the form

$$2i\, \partial E/\partial z + \partial^2 E/\partial x^2 + E\, f(|E|^2) = 0 \tag{1}$$

where f(I) is an arbitrary function of the intensity $I = |E|^2$ with $f(0) = 0$. When $f(I) = \alpha I$, ($\alpha = \text{const}$), Eq.(1) is the so called cubic nonlinear Schrödinger equation [6–8] (which corresponds to Kerr-nonlinearity in optical propagation). In the case of two-dimensional self-trapping, [6] z is a normalized axis of the soliton propagation and x is a normalized transversal axis (both of them are dimensionless and correspond to the real coordinate z and x multiplied by the wave number $k = \omega n/c$). The one-dimension pulse propagation along the z_1 axis in the slightly dispersive medium or optical fiber [8] with the group velocity $v = d\omega/dk$ and a nonlinearity $f_1(|E|^2)$, can be described by the same Eq.(1), where now $z = z_1 k^2\, dv/d\omega$, $x = k\, v\, (t - z_1/v)$, and $f = k\, f_1 /(dv/d\omega)$. In both of the cases f is proportional to the normalized nonlinear (i.e. intensity dependent) component $\Delta\epsilon^{NL}$ of the dielectric constant ϵ of medium. The first three invariants (i.e. quantities independent of z) of Eq.(1) are found [3] to be

$$J_1 = \int_{-\infty}^{\infty} I\, dx;\quad J_2 = i \int_{-\infty}^{\infty} (E E_x^* - \text{c.c.})\, dx;\quad J_3 = \int_{-\infty}^{\infty} [\,|E_x|^2 - \int_0^I f(s)\, ds]\, dx\,, \tag{1'}$$

corresponding to conservation of the total power, total "transverse" momentum and "transverse" energy of the field respectively.

The stationary solutions (in particular, singular solitons) of Eq.(1) have nonvarying intensity profile, $\partial|E|^2/\partial z = 0$, i.e. such solutions are written as

$$E(x, z) = u(x) \exp(i\delta z/2 + i\phi),$$

where u(x) is the real amplitude, ϕ is some real constant phase, and δ is an (unknown) real speed (or propagational constant) of the soliton. Thus, the equation for u(x) is:

$$d^2u/dx^2 + u[f(u^2) - \delta] = 0 \qquad (2)$$

whose soliton solution must satisfy the condition $u \to 0$ as $|x| \to \infty$ in order for the total power $P = \int_{-\infty}^{\infty} u^2 dx$ to be limited. This provides for the first integral of Eq. (2) in the form [1]

$$(du/dx)^2 = 2 \int_0^u u[\delta - f(u^2)]du \qquad (3)$$

integration of which determines the soliton amplitude profile u(x) for each particular δ and $f(u^2)$. The respective integrals can be analytically evaluated only for some particular functions $f(u^2)$ which may not be done in general case of arbitrary $f(u^2)$. In order to evaluate a total power P, however, one needs not to know an explicit form u(x). Indeed, by making use of Eq. (3) and bearing in mind that $u^2 = I$, one shows that [1]

$$P(\delta) = \int_0^{I_m(\delta)} dI/\sqrt{\delta - F(I)}, \qquad (4)$$

where $F(I) = I^{-1}\int_0^I f(I)\, dI$, i. e. P is determined immediately by f(I) and δ. In Eq (4), $I_m(\delta)$ is a peak intensity of the soliton; it is defined as a minimal positive root of the equation $F(I) = \delta$. The multistability of a single soliton arises when the function $\delta(P)$ implicitly determined by Eq (4) becomes multivalued (see Fig. 1).

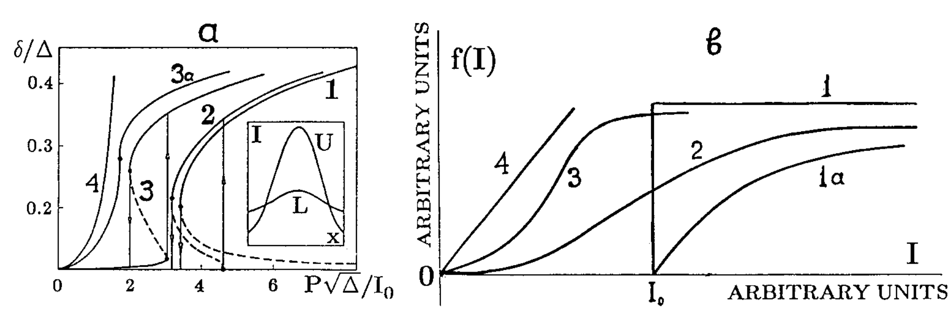

Fig. 1. A propagation constant δ versus the total power P carried by a single soliton (a) for different models of nonlinearity f as function of the field intensity I (b). "Step" function, Eq. (5), (curve 1 in (b)) and "smooth step" function, Eq. (7), (curve 1a in (b)) result in the same kind of behavior (curve 1 in (a)); the "square" model (i.e. $f(I) = o(I^2)$ as $I \to 0$) with saturation (curve 2 in (b)) results in the hysteretic jumps as in curve 2 in (a); the model, Eq. (8), (curve 3 in (b)) and "linear & smooth step" model, Eq. (9), results in truly hysteretic behavior, curve 3 in (a), with curve 3a in (a) corresponding to the critical relation between parameters of nonlinearity; Kerr-nonlinearity, f(I) proportional to I, (curve 4 in (b)) gives rise to nonhysteretic behavior (curve 4 in (a)). The broken lines at curves 1-3 in (a) correspond to the unstable solitons. In the insertion in (a), the intensity profiles I(x) are depicted of solitons that carry the same power but correspond to different branches of function $\delta(P)$ - upper branch (U) and lower branch (L).

It is readily shown that a Kerr-like nonlinearity ($f = \alpha I$), results only in a one-valued single soliton (with δ proportional to P^2); the same is valid also for any other nonlinearity with $f = \alpha I^{\mu}$ where $\mu > 0$ (but $\mu \neq 2$). In order to demonstrate existence of the countable set of states of the single soliton (with more than one state) we consider as the simplest example, the step-nonlinearity model [1]:

$$f(I) = 0, \text{ if } I < I_0 , \text{ and } f = \Delta , \text{ if } I > I_0, \tag{5}$$

where I_0 and Δ are some positive constants (curve 1 in Fig. 1b). Substituting Eq(5) into Eq(4) one obtains (curve 1 in Fig. 1a)

$$P(\delta) = \frac{I_0}{\sqrt{\Delta}} \frac{1}{1-\beta} \left(\frac{1}{\sqrt{\beta}} + \frac{\arcsin\sqrt{\beta}}{\sqrt{1-\beta}} \right); \quad \beta \equiv \frac{\delta}{\Delta} . \tag{6}$$

The function β vs P determined by Eq(6) is a two-valued function for any $P > P_{cr1} \approx 3.44 I_0/\sqrt{\Delta}$ with $\beta(P_{cr}) \approx 0.21$. The further example is given by the nonlinearity (curve 1a in Fig. 1b)

$$f(I) = 0 , \text{ if } I < I_0 \text{ and } f(I) = \Delta(1 - I_0^2/I^2) , \text{ if } I > I_0 . \tag{7}$$

In contrast with Eq(5), f(I) is now a continuous function, whereas its derivative df/dI is still discontinuous. The total power, Eq(4), now essentially represents the same kind of behavior as Eq(6), i.e. provides two-valued solution $\beta(P)$ for any $P > P_{cr2} \approx 4.28\, I_0/\sqrt{\Delta}$ with $\beta(P_{cr}) \approx 0.26$. In these cases, the nontrivial branches of the function $P(\delta)$ tend to infinity as $\delta \to 0$ and $\delta \to \Delta$ (note that the third, "trivial", branch with $\delta \equiv 0$, P - arbitrary, corresponds to a nontrapped beam with $I_m < I_0$), see curve 1 in Fig. 1a. This suggests a bistability without hysteresis and is due to the fact that nonlinearity f(I) differs from zero only for some finite $I > I_0$.

In order to attain truly hysteretic bistable behavior [i. e. that characterized by the S-shape steady-state curves of $\delta(P)$, see curve 3 in Fig. 1a, or N-shaped curves of $P(\delta)$, which causes both "on" and "off" jumps between different branches of the curve], the function f(I) must be positive at least in some range $0 < I < I_1$ and have a distinct peak of its first derivative df/dI in this range. The existence of hysteretic jumps is secured if $d\delta/dP = \infty$ (or $dP/d\delta = 0$) for two (or more) discrete values of P (or δ). A few such models of nonlinearity were proposed in [1], most of them having a Kerr-like behavior of f(I) as $I \to 0$ and "smooth-step" shape as I increases i.e. sharp increase beyond Kerr-like interval and then saturation plateau (see curve 3 in Fig. 1b; see also discussion of "linear & smooth step" model below). Such a nonlinear behavior may be originated, e. g. by the resonant multiphoton transitions [9] with saturation, or by the light-induced phase transition with a pronounced threshold (for example, in liquid crystals [10]). When such a nonlinearity can be represented in the vicinity I=0 by the expansion

$$F(I) = a_1 I + a_2 I^3 - a_3 I^5 + \ldots; \quad a_1, a_2, a_3 > 0; \tag{8}$$

the critical condition for the nonlinearity f (I) to give rise to the truly hysteretic soliton behavior (see e.g. curve 3 in Fig. 1a) is $a_1 a_3/a_2^2 \lesssim 0.08$.

Typically, the predicted number of single solitons for the fixed value of the total power carried by each one of them is odd; one may expect at least some of them to be unstable (e.g. one out of three). It was suggested in [1] that the solitons having a derivative $d\delta/dP < 0$ are unstable, whereas those with $d\delta/dP > 0$ are stable. This suggestion was later confirmed [2] by Enns and Rangnekar using a computer simulation for one of models of nonlinearity proposed in [1] [see here, Eq.(7)]. In this simulation, they subjected bistable solitary solutions corresponding to the nonhysteretic "smooth-step" model, Eq. 7, to a stringent stability test making them collide with each other; their results showed that indeed for that particular nonlinearity, the solitons with $d\delta/dP < 0$ are unstable and vice versa.

Further computer simulations [3] by Enns, Rangnekar, and Kaplan confirmed these results not only for the "smooth-step" model, Eq. 7, but also for other models resulting in truly hysteretic response 8(P). One of the most representative of these latter models is the so called "linear & smooth step" model [3] which corresponds to the splicing of well explored Kerr-nonlinearity for low intensity I into the "smooth-step" model, Eq. 7, i.e.

$$f(I) = \alpha I \text{ if } I \leq I_0 \text{ and } f = \Delta[1-(1-\mu)I_0^2/I^2] \text{ if } I \geq I_o , \tag{9}$$

where $\alpha, \mu > 0$, which allows for existence of three-value hysteretic curve $\delta(U)$ if $\mu \gtrsim 0.42$. Stability and instability of different solitons attributable to this model is demonstrated in Fig. 2, in which the results of pair collision of two solitons are shown in the cases when the solitons belong to the lower and upper branches of curver $\delta(P)$ (and therefore both of them are stable, Fig. 2a), the lower and middle branches (the former one is stable and the latter - unstable, Fig. 2b), and when both of them belong to the same stable branch (in this case, to the upper one) and therefore, both are stable (Fig. 2c).

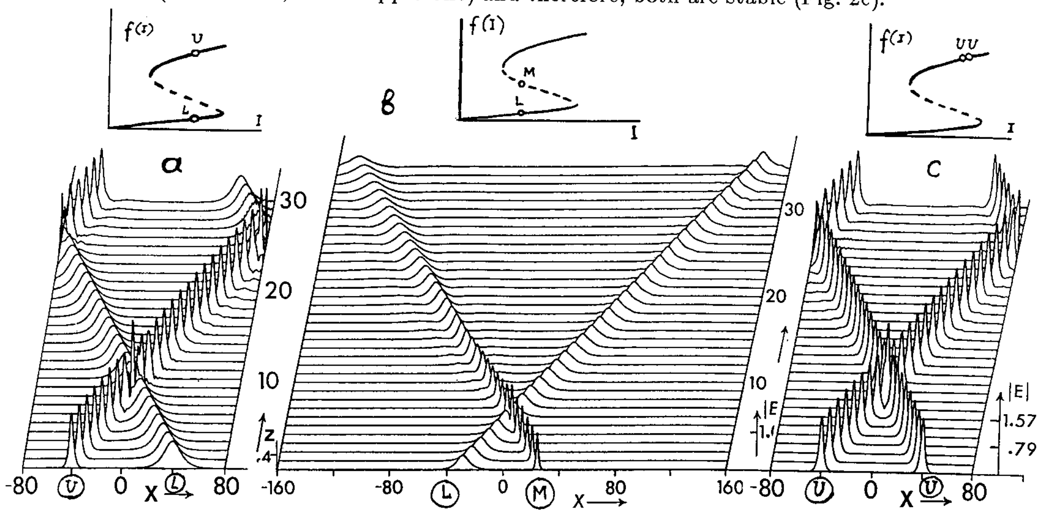

Fig. 2. Collisions between: two solitons corresponding to the lower, L (stable), and upper, U (unstable), branches of hysteretic curve with both solitons surviving the collision (a); two solitons corresponding to the lower, L, and middle, M (unstable), branches with the surviving L soliton and decaying M soliton (b); two identical solitons corresponding both to the upper, U, (stable) branch with both soliton stable surviving the collision (c).

The stable solutions of the linear & smooth step model, Eq. (9), turns out to be absolutely stable solitons (the so called robust solitons, see below), i.e. stable not only against small perturbations, but also against such large perturbations as collisions with other solitons. However, this situation turns out to be not valid for arbitrary nonlinearities; it was found [3] that nonlinearities exist that give rise to the solitons which are "weakly" stable (i.e. stable only against sufficiently small perturbations). The detailed search [3] done for a wide variety of models of nonlinearity most of which were "designed" to exhibit multistate solitary solutions, revealed a few very distinct facts about stability/instability of new solitons/solitary solutions:

(i) The solitary solutions with $d\delta/dP < 0$ are unconditionally unstable (i.e. they decay under action of even very small perturbation, such as, e.g., computer numerical noise).

(ii) In the general case of arbitrary nonlinearity (at least in the class of all models of nonlinearity studied in [3]), the relationship $d\delta/dP > 0$ is found to be a necessary but not sufficient condition of absolute stability of solitons. On the other hand, it was shown that it is a sufficient condition for "weak" stability, i.e. stability of a soliton against a sufficiently (but not infinitesimally) small perturbation. It is important though to note that this small perturbation can be a collision with the other soliton having sufficiently small peak amplitude. Therefore, in the framework of "weak" stability, one still can have two solitons which perfectly survive their collision. These solutions were called [3] "weak" solitons.

(iii) For some class of models of nonlinearity, the condition $d\delta/dP > 0$ remains a criterion of absolute stability of solitary solutions (and therefore, a criterion that these solutions are indeed, solitons); i. e. they are stable against any (including large) perturbations, in particular, in the form of collision with any other solitary solution. In order to describe these absolutely stable solitary solutions, the notion of "robust" solitons [3] was introduced. We showed also that bistable robust solitons are possible.

(iv) By studying numerous models of nonlinearity, we suggested (in addition to the condition $d\delta/dP > 0$) the following criteria of "robustness":
(a) $f(I)/I^2 = o(1)$ as $I \to \infty$, the condition for stability against collapse (in other terms, explosive behavior or self-focusing singularities).
(b) f(I) is a non-negative and nondecreasing function for $I > 0$; this is a sufficient condition for stability against dispersion,

The significance of existence of two types of soliton stability and, respectively, weak and robust solitons, should not be underestimated. Solitons, by definition, are stable solitary solutions of nonlinear wave equations. These equations and their solitons result from many problems [7] in the theory of elementary particles, nonlinear optics and electrodynamics, plasma physics, hydrodynamics, biology, etc. It is well known, for example, that the non-degenerate solitary wave solutions of the cubic-nonlinear Schrödinger equation (which has many applications in nonlinear optics) are stable against both small and large perturbations; in particular, two such singular solitary waves survive their collision, with their individual energies and momentae conserved after the collision [6], i.e., they are solitons. Since the cubic-nonlinear equation is the most famous of the nonlinear Schrödinger-like equations studied so far, the distinction between these two types of stability has never been, to the best of our knowledge, clearly drawn in the literature. Very often for such (and other) [11] nonlinear equations, the conditions of stability from small-perturbation analysis are automatically regarded as universal criteria for soliton existence. Our results [3] showed, however, that the stability against small perturbations alone (as well as instability against large perturbations alone) does not provide a comprehensive description of stability of solitary wave solutions of highly-nonlinear Schrödinger equations and require the notion of weak and robust solitons. The fact that "robust" bistable solitons of the highly-nonlinear Schrödinger equation are possible may prove to be of significant importance for such nonlinear optical applications as fiber optics communication with undistorted pulses [8], optical switching and bistability [11] and stimulate the search for the materials and mechanisms with appropriate nonlinearities.

Soliton bistability may provide new opportunities in the field of optical bistability. Indeed, for example, a bistable self-trapping of light provides a potential for optical bistable device entirely free either from any cavities, nonlinear interfaces, nonlinear waveguides, four-wave mixing, etc. Probably most importantly, bistable soliton pulses in nonlinear fiber waveguides with an appropriate nonlinearity may provide the first (to the best of our knowledge) known opportunity to attain a temporal (or dynamic) bistability as opposed to all known kinds of optical bistability which were so far formulated in terms of steady-state regimes. [12] The very notion of steady-state optical bistability comes into the inevitable contradiction with the applications most of which assume fast pulse regime of operations. The truly dynamic (or temporal) bistability discussed here is based on bistable pulse shapes (as well as on bistable duration of the pulses) and offers a way to resolve this contradiction.

Recent computer simulations [4] by Enns and Rangnekar explicitly demonstrated that switching between robust bistable soliton states can occur under action of external signal; this is very encouraging fact for the potential applications of bistable solitons. Most recently, by using Painleve test Enns [5] showed that the nonlinear Schrödinger equation with at least one particular models of nonlinearity [3] giving rise to robust bistable solitons, is ***completely integrable.*** This suggests a direct mathematic prove that the objects in consideration are indeed solitons even under strict "integratability" requirement (although the known definitions [7] of solitons do not even stipulate such a requirement explicitly).

The bistable solitons may exist also in the case of three-dimensional propagation. [1] In order to attain a hysteretic bistable soliton propagation the lower required degree of nonlinearity at $I \to 0$ is f proportional to I^3 [with f attaining some maximum or saturation as I increases, e. g. $f = \alpha I^3/(1+I^3/I_0^3)$]. Such a nonlinearity can be originated, e. g. by the three-photon resonant absorption.

In conclusion, we have demonstrated an existence of multi-stable solitons of highly- nonlinear Schrödinger equation. In order for those solutions to exist, the nonlinearity must satisfy some special conditions, e.g. its dependence on the light intensity must have a range where it increases sufficiently sharply. We have, further, demonstrated the necessity to distinguish between stability against small and large perturbations, introduced the concept of "robustness" for solitons and suggested a criterion for "robustness," and most importantly, shown that bistable "robust" solitons are possible. In nonlinear optics, these solitons may manifest themselves either as singular pulses (e.g. in nonlinear fibers) or self-trapped channels (both in two- and three-dimension cases). Bistable solitons present the ultimate case of multistable wave propagation and may find an application to the dynamic (temporal) optical bistability.

I greatly appreciate numerous and fruitful interactions with R. H. Enns and S. S. Rangnekar. This work was supported by the US Air Force Office of Scientific Research.

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