Basics of Wave and Quantum Mechanics for Engineers
ECE 520.457, Fall 2007

Class Notes

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4.3. Diffraction; "paraxial" approximation and Gaussian beams.

In general, the 3D-propagation of EM-waves could be a tough cookie from mathematical point of view, even for a monochromatic wave, (4.11). However, some of the major phenomena, like diffraction of a light beam with a smooth transverse profile, can readily be analyzed analytically in a closed form that provides us with very simple and transparent formulas.

The 1D-case (string-like), corresponds to the simplest wave motion, that translates into the behavior of strictly plane wave that have an infinite phase front (in its transverse cross-section) and propagates along the straight line with its (plane) profile staying intact. But only 2D or 3D-cases (i. e. when the wave has a finite profile in its cross-section) reveal one of the most fundamental properties of wave propagation: such a wave cannot keep its profile while propagating in a homogeneous medium (e. g. in vacuum); it cannot be contained within any limited size. Regardless of the phase/amplitude of its initial shape/profile, after a certain propagation distance, such a wave (light ***beam***) will start spreading, so that at sufficiently long distance we will see a ***light cone*** fanning away from the main axis of propagation. This phenomenon is called a ***diffraction***, and is typical for any non-plane wave. The scale distance along the axis of propagation for this to happen is called ***diffraction*** length, l_D or ***Rayleigh*** parameter, and the cone angle is called ***angle of diffraction***, ϕ_D. Both of them are directly related to the ratio of the wavelength, λ, and the initial beam size, a; as one expects, the smaller is this ratio, the smaller the diffraction. As we will see below,

$$\phi_D = O\left(\frac{\lambda}{a}\right); \quad and \quad \frac{l_D}{a} = O\left(\frac{a}{\lambda}\right) \tag{4.12}$$

The general theory of diffraction is the heck of mathematics, and it can be found in most of "thick" books on classical optics, which traditionally address subjects like wave scattering on an infinitely sharp single edge or slit. This kind of processes involve as large perturbations of plane wave as one can imagine. However, as one might suspect, a very smooth wave profile (or a smooth "scatterer"), with its transverse scale, a, being much larger than λ, should behave pretty close to plane wave at each point along the main axis of propagation (we will call it ***quasi − plane*** wave): should not it help to make a greatly simplifying approximation, which in turn may help to get simple results? Besides, most of the currently known lasers typically emit (in a very natural way) precisely this kind of output beams. This approximation is based on the fact that the light beam is broad, and thus diffraction angle is small,

$$\lambda \ll a\,, \quad \phi_D \ll 1 \tag{4.13}$$

hence the so called ***paraxial*** approximation (i. e. the one addressing the events at small angles near the axis); also ***quasi – optical***, ***broad-beam***, etc.

Assume now that the z axis is the main axis of propagation, a ***quasi-plane*** wave travels along it. Definition of ***quasi-plane*** wave: the amplitude profile $|u|$ as well as the phase profile $\phi = -i\,ln(u/|u|)$ are ***slowly*** varying functions as compared to e^{ikr} in any cross-action ***normal to z***. Therefore, these waves are varying even ***more*** slowly (compared to e^{ikz}) along z axis. Assume also that $k^2 = \omega^2\varepsilon/c^2$ is either constant or ***varying slowly*** along the z axis, and its variations are ***small*** $(|\delta k| \ll k_0)$ i.e. $\varepsilon = \varepsilon_0 + \delta\varepsilon$; with $|\delta\varepsilon\,(\vec{r})| \ll \varepsilon_0$, where ε_0 is the dielectric constant of unperturbed homogeneous medium, and the small variation $\delta\varepsilon\,(\vec{r})$ can be due to inhomogeneity of material if any. We express now the field in the following form:

$$\tilde{u} = \upsilon e^{ik_0 z} \quad with \quad k_0^2 = \frac{\omega^2\varepsilon_0}{c^2} \tag{4.14}$$

where $e^{ik_0 z}$ is an unperturbed plane wave, plug it into Eq. (4.11) and evaluate $\partial^2\tilde{u}/\partial z^2$:

$$\frac{\partial^2\tilde{u}}{\partial z^2} = \left(\frac{\partial^2\upsilon}{\partial z^2} + 2ik_0\,\frac{\partial\upsilon}{\partial z} - {k_0}^2\upsilon\right)e^{ik_0 z} \tag{4.15}$$

Here υ is a complex amplitude profile of the wave; it is varying very slowly along z axis. Therefore it is OK for us to neglect at least $\partial^2\upsilon/\partial z^2$ in (4.15). Substituting (4.15) with this term thrown out, into (4.11) we get:

$$2ik_0\,\frac{\partial\upsilon}{\partial z} + \Delta_\perp\upsilon + \frac{\delta\varepsilon(\vec{r})}{\varepsilon_0}\,k_0^2\,\upsilon = 0 \tag{4.16}$$

where $\Delta_\perp = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}$ is a "transverse Laplacian". This is a ***paraxial*** equation we were looking for. It is of the ***first – order*** in z, and is usually much easier to solve this equation rather than the original "full" wave equation, Eq. (4.11). The paraxial equation bridges general theory of propagation with the approximation of ***ray*** optics (or ***geometric*** optics) in that it gives (somewhat simplified) diffraction picture of propagation being still easily transformed into geometric optics. A great majority of wave propagation processes in modern optics (and microwaves) can be described by Eq. (4.16), e.g. optical waves inside and outside laser, diffraction in various systems, propagation in systems with varying parameters, or in inhomogeneous media, e. g. propagation in waveguides and optical fibers.

The amazing fact as we will see further on, is that this equation is essentially the same the central equation of* quantum mechanics*, the Schrӧdinger equation for the wavefunction ψ! The latter one is obtained from the EM paraxial equation (4.16) by replacing field υ with ψ, k_0 with a constant proportional to $\hbar$, "slow" axis z with the **time** ***t, and $\delta\varepsilon(\vec{r})$ with the potential $U(\vec{r})$. They imply also similar "limit" physics: while the transition $k_0 \to 0$ in***

paraxial approximation corresponds to the limit of ray optics, the transition $\hbar \to 0$ ***in the Schrödinger*** equation corresponds to the limit of classical mechanics.

For a ***homogeneous*** media, i.e. when $\delta\varepsilon = 0$, (4.16) is reduced to

$$2ik_0 \frac{\partial \upsilon}{\partial z} + \Delta_\perp \upsilon = 0 \qquad (4.17)$$

The general solution of this equation may be represented in many different ways (e.g. again, by the ensemble of plane waves); however, the most distinguished and most important for applications (as well as for understanding of diffraction phenomenon) is extension of the solution into the so called ***Gaussian modes***. Each of these modes is the so called ***auto-model*** solution in that these solutions ***conserve*** the mathematical "shape" of their amplitude and phase ***profiles*** during their propagation in the z-axis (whereas ***parameters*** of the profile: radius of the beam, its maximum amplitude, and curvature of the phase front, are changing). The lowest mode at the beam "waist" (see below) has the simplest Gaussian profile

$$|\upsilon| \propto e^{-r^2/2\rho^2} \qquad (4.18)$$

where $r = \sqrt{x^2 + y^2}$ is the distance from the center of the beam to the point of observation. We will now assume a ***cylindrically – symmetric*** beam, such that

$$\Delta_\perp = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} \qquad (4.19)$$

(a symmetric beam is independent on the angle in the x, y plane); therefore

$$2ik_0 \frac{\partial \upsilon}{\partial z} + \frac{\partial^2 \upsilon}{\partial r^2} + \frac{1}{r}\frac{\partial \upsilon}{\partial r} = 0 \qquad (4.20)$$

Now, we make a guess, that the lowest order auto-model solution has the form

$$\upsilon(z, r) = A(z) \exp\left[-\xi(z) r^2/2\right] \qquad (4.21)$$

where the yet unknown characteristics of solution, A and ξ (both complex), are assumed to be the functions of ***only*** the distance of propagation, z! Our bet pays immediately, for we obtain ***ordinary*** differential equations for A and ξ:

$$ik_0 A' = A\xi\ , \quad ik_0 \xi' = \xi^2 \qquad (4.22)$$

(where “prime" denotes d/dz), which are readily solved as:

$$\xi = \frac{k_0}{iz + c_1}\ ; \quad A = \frac{c_2}{iz + c_1} \qquad (4.23)$$

(c_1 and c_2 - constants). Here $A(z)$ a (complex) ***amplitude*** of the beam, i. e. $|A(z)|$ is a maximal amplitude of the profile in its cross-section, whereas for ξ we have

$$Re(\xi) = \frac{1}{\rho^2}\ ; \quad Im(\xi) = \frac{k_0}{R} \qquad (4.24)$$

where $\rho(z)$ is the size of the beam, and R is the radius of curvature of phase front of the beam.

Thus, the beam has a spherical wave front at any z. Suppose, at some $z = z_0$ the phase front is ***flat***, i.e. $R = \infty$. This point is called the ***waist*** of the beam, because at it the beam size (i. e. the distance in the cross-section, where $\upsilon = A exp(-1/2)$) is ***minimal***, $\rho = \rho_0 \equiv a$. At that point in z, the amplitude has its ***maximum***, $A = A_0$.

before we write down the solution (4.21) in explicit form with ξ and A solved for the conditions at waist. we introduce a characteristic distance of the beam divergence (which is also called a diffraction distance, or ***Rayleigh*** or confocal parameter) for the beam with the size a:

$$l_D = k_0 a^2 \tag{4.25}$$

and normalize the distance from the waist, $z - z_0$ by this parameter by introducing a dimensionless distance, ζ:

$$\zeta = (z - z_0)/l_D \tag{4.26}$$

In terms of ζ the solution (4.23) that satisfy the condition $A = A_0$, $R = \infty$ and $\rho = a$ at $z = z_0$, is

$$\frac{A}{A_0} = a^2 \xi(\zeta) = Z(\zeta)\,, \quad with \quad Z(\zeta) \equiv \frac{1}{1 + i\zeta} \tag{4.27}$$

and thus the full solution (4.21) is

$$\upsilon(\zeta\ , r) = A_0 Z(\zeta)\ e^{-\frac{Z(\zeta) r^2}{2a^2}}\ ; \tag{4.28}$$

Using (4.24), we have now the spatial dynamics of all the characteristics of the diffracting beam:

$$\frac{|A|^2}{|A_0|^2} = \frac{1}{1 + \zeta^2}\,, \qquad \frac{\rho}{a} = \sqrt{1 + \zeta^2}\,; \qquad and \qquad \frac{R}{l_D} = \zeta + \frac{1}{\zeta} \tag{4.29}$$

We can see that as $z \to \infty$, or at least as $\zeta \gg 1$ (the so called $far - field$ area), the size of beam increases almost linearly with the distance:

$$\rho \approx (z - z_0)/k_0 a \tag{4.30}$$

which determines the linear divergence of the beam in the far-field area. This divergence is characterized by a ***diffraction*** angle as

$$\phi_D = \frac{\rho(z \to \infty)}{z - z_0} = \frac{1}{k_0 a} \ll 1\,. \tag{4.31}$$