Basics of Wave and Quantum Mechanics for Engineers

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Solution for the general uncertainty relationship (6.55)

We introduce a ***commutator***, (6.51), as:

$$[\hat{A}, \hat{B}] \equiv \hat{A}\hat{B} - \hat{B}\hat{A} \equiv i\hat{C}, \qquad (1)$$

where $\hat{C}$ is the ***remainder of commutation***. We say that two operators ***commute***, if they have property $\hat{C}=0$, and they do ***not*** commute, if $\hat{C}\neq 0$. The commuting operators corresponds to quantities that can be measured simultaneously with arbitrary precision, e. g. the product $\delta x \delta y$ does not have any bottom limit, while the product $[\hat{x}, \hat{p}_x]$ does.

We look for an uncertainty relation in general case of $\hat{A}$ and $\hat{B}$ (and thus $\hat{C}$) being Hermitian operators (thus having $<A>$, $<B>$, and $<C>$ all real). The operators for the ***deviation*** from the respective mean values (or mathematical expectations) are (6.53):

$$\Delta\hat{A} \equiv \hat{A} - <A>; \quad \text{and} \quad \Delta\hat{B} \equiv \hat{B} - <B>; \quad \textit{with} \quad <\xi> = \int \psi^* \hat{\xi} \psi dx. \qquad (2)$$

The uncertainty δA for the quantity A is defined then as (6.54):

$$\delta A \equiv \sqrt{<\Delta\hat{A}^2>} = \sqrt{<(A - <A>)^2>} = \sqrt{<A^2> - <A>^2}; \qquad (3)$$

where $<\xi^2> = \int \psi^* \hat{\xi}^2 \psi dx$. We want to show that (6.55):

$$\delta A \cdot \delta B \geq <C>/2, \qquad (4)$$

which is the generalized uncertainty relation. To that end we introduce the so called the ***inner product of two functions***, $f(x)$ and $g(x)$ and use ***Dirac*** "brackets":

$$<f|g> = \int_a^b f(x)^* g(x) dx = <g|f>^* \qquad (5)$$

and in particular,

$$<f|f> = \int_a^b |f(x)|^2 dx \quad \text{and} \quad <g|g> = \int_a^b |g(x)|^2 dx \qquad (6)$$

and will be using the so called ***integral Schwarz inequality***:

$$\sqrt{<f|f><g|g>} \geq |<f|g>| \qquad (7)$$

(For a proof, one can see e. g. F. Riesz and B. Sz-Nagy, ***Functional Analysis***, Unger, New York, 1955, section 21.) We designate now:

$$f(x) = (\Delta\hat{A})\psi \quad \text{and} \quad g(x) = (\Delta\hat{B})\psi \tag{8}$$

so that

$$\delta A = \sqrt{<f|f>}, \quad \text{and} \quad \delta B = \sqrt{<g|g>} \tag{9}$$

Thus, multiplying these two and using the Schwarz inequality, (7), we have:

$$\delta A \cdot \delta B \geq |<f|g>| \tag{10}$$

We notice now that for any imaginary z we can write:

$$|z|^2 = [\mathrm{Re}(z)]^2 + [\mathrm{Im}(z)]^2 \geq [\mathrm{Im}(z)]^2 = [(z - z^*)/2i]^2 \tag{11}$$

By letting $z = <f|g>$, and recalling the definition of $<f|g>$, via (6), (8), and (2), we can prove that in our case,

$$<f|g> = <\hat{A}\hat{B}> - <\hat{A}> <\hat{B}> \quad \text{and} \tag{12}$$

$$<g|f> = <f|g>^* = <\hat{B}\hat{A}> - <\hat{A}> <\hat{B}>$$

so that

$$<f|g> - <g|f> = <\hat{A}\hat{B}> - <\hat{B}\hat{A}> = <[\hat{A},\hat{B}]> = i<C> \tag{13}$$

Returning to (11), we have $|<f|g>| = <C>/2$. Using this in Eq. (10), we obtain Eq. (4), which we were seeking to prove.